

DECOMPOSITION OF ACYCLIC NORMAL CURRENTS IN A METRIC SPACE

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ABSTRACT. We prove that every acyclic normal one-dimensional real Ambrosio-Kirchheim current in a Polish (i.e. complete separable metric) space can be decomposed in curves, thus generalizing the analogous classical result proven by S. Smirnov in Euclidean space setting. The same assertion is true for every complete metric space under a suitable set-theoretic assumption.

1. INTRODUCTION

The main result of the paper is Theorem 5.1 which says, very roughly speaking, that every acyclic normal one-dimensional real current in a complete metric space can be decomposed in curves. By currents here we mean Ambrosio-Kirchheim currents introduced in [1] and generalizing the classical Whitney flat chains in Euclidean space from [5]. For the readers' convenience we recall some basic facts about Ambrosio-Kirchheim currents in Appendix B.

Throughout the paper we silently assume, as it is now customary when dealing with metric currents, that the density character (i.e. the minimum cardinality of a dense subset) of every metric space is an Ulam number. This guarantees that every finite positive Borel measure is tight (even Radon when the space is complete), is concentrated on some σ -compact subset and the support of this measure is separable (see, e.g., proposition 7.2.10 from [4]). Besides being consistent with the Zermelo-Fraenkel set theory, this assumption is also not restrictive because, as mentioned in [1], the whole theory of metric currents could have been developed under the additional requirement that mass measures of the currents be tight. In fact, without this assumption our result will be proven to hold for every complete metric space when μ_T and $\mu_{\partial T}$ are tight, and hence, in particular, for Polish (i.e. complete separable metric) spaces.

In the Euclidean space setting the analogous result on decomposition of acyclic normal currents in curves has been first proven by S. Smirnov (see theorem C from [6]) and further applied in many papers, especially dealing with optimal mass transportation problem. Our result thus generalizes the classical one from [6] to generic metric spaces and hence opens the way to new treatment of optimal mass transportation problems in different metric structures. The technique of the proof we adopt is different from the one used in [6] and hence is new also in Euclidean setting.

As an illustration of the results of the paper, in Appendix A we study a formulation of the optimal mass transportation problem in terms of metric currents and show that in most reasonable cases of geodesic metric spaces it is equivalent to the classical Monge-Kantorovich setting, while in general it is not, and, moreover, from an applicative point of view it is more natural for mass transportation. Although this can be proven also by other means, we think that the use of decomposition

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result for metric currents is the most natural and easy way to get it, and, moreover, such a strategy is helpful also for different kind of optimal transportation problems.

2. NOTATION AND PRELIMINARIES

For metric spaces X and Y we denote by $\text{Lip}(X, Y)$ (resp. $\text{Lip}_b(X, Y)$) the set of all (resp. bounded) Lipschitz maps $f: X \rightarrow Y$ with arbitrary Lipschitz constant. If $Y = \mathbb{R}$, we will omit the reference to Y and write simply $\text{Lip}(X)$ and $\text{Lip}_b(X)$ respectively. The Lipschitz constant of a map $f: X \rightarrow Y$ will be denoted by $\text{Lip}(f)$. The supremum norm of a map $f: X \rightarrow \mathbb{R}$ is denoted by $\|f\|_\infty$.

The metric spaces considered will further be tacitly assumed to be complete, unless explicitly stated otherwise.

All the measures we will consider in the sequel are signed Borel measures with finite total variation over some metric space E . The narrow topology on measures is defined by duality with the space $C_b(E)$ of continuous bounded functions. For a set $e \subset E$ we denote by $1_e: E \rightarrow \mathbb{R}$ its characteristic function.

We recall that a Banach space E is said to have *bounded approximation property* whenever for every compact set $K \subset E$ there is a sequence of linear operators $\{T_n\}$, $T_n: E \rightarrow E$, of finite rank (i.e. with finite-dimensional images), such that $\|T_n x - x\|_E \rightarrow 0$ for all $x \in K$ as $n \rightarrow \infty$, and the operator norms of T_n are bounded by a universal constant $C > 0$. If one can choose this sequence so as to have $C = 1$ then one says that E has *metric approximation property*. Clearly, the above convergence is uniform in K . In fact, if $\{y_\nu\} \subset K$, $y_\nu \rightarrow y$ as $\nu \rightarrow \infty$, then

$$\|T_\nu y_\nu - y\|_E \leq \|T_\nu y_\nu - T_\nu y\|_E + \|T_\nu y - y\|_E \leq C\|y_\nu - y\| + \|T_\nu y - y\|_E \rightarrow 0$$

as $\nu \rightarrow \infty$. A typical example of Banach spaces with bounded approximation property is given by Banach spaces possessing Schauder (topological) basis.

2.1. Curves. We equip the set of Lipschitz curves $\theta: [0, 1] \rightarrow E$ with the distance (2.1)

$$d_\Theta(\theta_1, \theta_2) := \inf \left\{ \max_{t \in [0, 1]} d(\theta_1(t), \theta_2(\phi(t))) : \phi: [0, 1] \rightarrow [0, 1] \text{ bijective increasing} \right\},$$

and call two Lipschitz-continuous curves $\theta_1, \theta_2: [0, 1] \rightarrow E$ *equivalent*, if

$$d_\Theta(\theta_1, \theta_2) = 0.$$

It is not difficult to show that the equivalence of θ_1 and θ_2 means the existence of continuous surjective nondecreasing functions (called usually “reparameterizations”) $\phi_1, \phi_2: [0, 1] \rightarrow [0, 1]$ such that $\theta_1(\phi_1(t)) = \theta_2(\phi_2(t))$ for all $t \in [0, 1]$. The set of equivalence classes of Lipschitz curves equipped with the distance d_Θ will be denoted $\Theta(E)$ (we will further usually omit the reference to E if it is clear from the context, and write simply Θ). In this way each $\theta \in \Theta$ can be clearly identified with some oriented rectifiable curve. In the sequel we will frequently slightly abuse the language, identifying the elements of Θ (i.e. oriented rectifiable curves) with their parameterizations (i.e. Lipschitz-continuous paths parameterizing such curves), when it cannot lead to a confusion. It is easy to see that $\theta_\nu \rightarrow \theta$ in Θ implies the Hausdorff convergence of the respective traces, though the converse is clearly not true.

We call $\theta \in \Theta$ an *arc*, if it is injective.

2.2. Ascoli-Arzelà theorem revisited. We will need the following version of an Ascoli-Arzelà type theorem.

Proposition 2.1. *Let E be a complete metric space, $\theta_\nu: [0, 1] \rightarrow E$ be Lipschitz functions with uniformly bounded Lipschitz constants $\text{Lip} \theta_\nu \leq L$ and satisfying the following uniform tightness condition: for every $\varepsilon > 0$ there is a compact set*

$K_\varepsilon \subset E$ such that $\mathcal{L}^1(\theta_\nu^{-1}(K_\varepsilon^c)) \leq \varepsilon$ for all $\nu \in \mathbb{N}$. Then there is a subsequence of θ_ν uniformly converging to some Lipschitz function $\theta: [0, 1] \rightarrow E$.

Proof. We prove first the statement in the case when E is a Banach space with bounded approximation property. In this case, given an $\varepsilon > 0$, consider a compact set $K \subset E$ such that $\mathcal{L}^1(\theta_\nu^{-1}(K^c)) \leq \varepsilon/8L$ for all $\nu \in \mathbb{N}$, and a linear finite rank operator T with operator norm $C > 0$ such that

$$\sup_{x \in K} \|x - Tx\| \leq \varepsilon/6.$$

Denoting $\theta'_\nu(t) := T\theta_\nu(t)$, one has that $\theta'_\nu: [0, 1] \rightarrow E_n$ are CL -Lipschitz functions with values in a finite dimensional subspace $E_n \subset E$. Since $\theta'_\nu([0, 1]) \cap TK \neq \emptyset$ for all $\nu \in \mathbb{N}$, then all θ'_ν are uniformly bounded, and hence by Ascoli-Arzelà theorem there is a subsequence of ν (which we do not relabel for brevity) such that θ'_ν are uniformly convergent. Let then $N \in \mathbb{N}$ be such that

$$\sup_{t \in [0, 1]} \|\theta'_k(t) - \theta'_\nu(t)\| \leq \varepsilon/6$$

for all $k \geq N$ and $\nu \geq N$. Thus for such k and ν we get

$$\|\theta_k(t) - \theta_\nu(t)\| \leq \|\theta_k(t) - \theta'_k(t)\| + \|\theta'_k(t) - \theta'_\nu(t)\| + \|\theta'_\nu(t) - \theta_\nu(t)\| \leq 3\varepsilon/6 = \varepsilon/2$$

whenever $t \in \theta_\nu^{-1}(K) \cap \theta_k^{-1}(K)$. Minding that

$$\begin{aligned} \mathcal{L}^1\left((\theta_\nu^{-1}(K) \cap \theta_k^{-1}(K))^c\right) &= \mathcal{L}^1(\theta_\nu^{-1}(K^c) \cup \theta_k^{-1}(K^c)) \\ &\leq \mathcal{L}^1(\theta_\nu^{-1}(K^c)) + \mathcal{L}^1(\theta_k^{-1}(K^c)) \leq \varepsilon/4L, \end{aligned}$$

we obtain that for every $s \in [0, 1]$ there is a $t \in \theta_\nu^{-1}(K) \cap \theta_k^{-1}(K)$ such that $|t - s| \leq \varepsilon/4L$. Therefore,

$$\begin{aligned} \|\theta_k(s) - \theta_\nu(s)\| &\leq \|\theta_k(s) - \theta_k(t)\| + \|\theta_k(t) - \theta_\nu(t)\| + \|\theta_\nu(t) - \theta_\nu(s)\| \\ &\leq L|t - s| + \varepsilon/2 + L|t - s| \leq \varepsilon \end{aligned}$$

for all $k \geq N$ and $\nu \geq N$. We have shown therefore that the chosen subsequence of θ_ν is uniformly Cauchy, hence uniformly converging to a L -lipschitz function $\theta: [0, 1] \rightarrow E$ as claimed.

For the case when E is a complete separable metric space, recall that by [2] there is a bi-Lipschitz embedding $g: E \rightarrow c_0$, where $c_0 \subset \ell^\infty$ stands for the Banach space of vanishing sequences, which possesses the Schauder basis and hence satisfies the bounded approximation property. It suffices then to apply the proven result to the sequence $g \circ \theta_\nu: [0, 1] \rightarrow c_0$, obtaining that a subsequence (not relabeled) of $\{g \circ \theta_\nu\}$ is uniformly Cauchy, hence so is the sequence $\{\theta_\nu\}$ (because g^{-1} is Lipschitz), and thus the latter converges uniformly to some L -Lipschitz function.

Finally, if E is a generic complete metric space (not necessarily separable), we just recall that $\cup_\nu \theta_\nu([0, 1])$ is σ -compact, hence separable, and we may consider θ_ν as acting into the closure $\overline{\cup_\nu \theta_\nu([0, 1])}$ of the latter, and refer to the above proven case. \square

3. SUBCURRENTS

In the sequel we will be frequently using the notion of a subcurrent of a given current as introduced in the definition below.

Definition 3.1. We say that S is a subcurrent of T , and write $S \leq T$, where T and S are k -dimensional currents, whenever

$$\mathbb{M}(T - S) + \mathbb{M}(S) \leq \mathbb{M}(T).$$

We now provide a series of remarks concerning the above definition.

Remark 3.2. Since the inequality

$$\mathbb{M}(T - S) + \mathbb{M}(S) \geq \mathbb{M}(T)$$

always holds true, then S is a subcurrent of T , if and only if the equality actually holds.

Remark 3.3. If $R \leq S$ and $S \leq T$, then $R \leq T$. In fact,

$$\begin{aligned} \mathbb{M}(T) &\geq \mathbb{M}(S) + \mathbb{M}(T - S) \geq \mathbb{M}(R) + \mathbb{M}(S - R) + \mathbb{M}(T - S) \\ &\geq \mathbb{M}(R) + \mathbb{M}(T - R), \end{aligned}$$

because of the triangle inequality $\mathbb{M}(T - R) \leq \mathbb{M}(T - S) + \mathbb{M}(S - R)$.

Remark 3.4. Let T be a current and let $e \subset E$ be a Borel set. Then $T \llcorner e \leq T$. In fact,

$$\mathbb{M}(T) = \mu_T(E) = \mu_T(e) + \mu_T(e^c) = \mathbb{M}(T \llcorner e) + \mathbb{M}(T - T \llcorner e).$$

Remark 3.5. If $S \leq T$, then for every Borel set $e \subset E$ one has $S \llcorner e \leq T \llcorner e$. In fact, by the triangle inequality

$$\begin{aligned} \mathbb{M}(T \llcorner e) &\leq \mathbb{M}((T - S) \llcorner e) + \mathbb{M}(S \llcorner e) \\ \mathbb{M}(T \llcorner e^c) &\leq \mathbb{M}((T - S) \llcorner e^c) + \mathbb{M}(S \llcorner e^c), \end{aligned}$$

while if we sum the above inequalities, then as a result we get an equality since $S \leq T$. Hence the above inequalities are in fact equalities for all Borel $e \subset E$. In particular, this also implies

$$(3.1) \quad \mu_T = \mu_{T-S} + \mu_S,$$

and hence $\mu_S \leq \mu_T$. On the other hand, if (3.1) holds, then $S \leq T$ since

$$\mathbb{M}(S) + \mathbb{M}(T - S) = \mu_S(E) + \mu_{T-S}(E) = \mu_T(E) = \mathbb{M}(T).$$

Lemma 3.6. *Let T_ν be a sequence of currents, $S_\nu \leq T_\nu$, and suppose that both $S_\nu \rightarrow S$ and $T_\nu \rightarrow T$ weakly as currents as $\nu \rightarrow \infty$, while $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$. Then $S \leq T$ and $\mathbb{M}(S_\nu) \rightarrow \mathbb{M}(S)$.*

Proof. Consider the sequence $\{T_\nu - S_\nu\}$ which converges to $T - S$ in the weak sense of currents. By the lower semicontinuity of \mathbb{M} we know that

$$\begin{aligned} \mathbb{M}(S) + \mathbb{M}(T - S) &\leq \liminf_{k \rightarrow \infty} \mathbb{M}(S_\nu) + \liminf_{k \rightarrow \infty} \mathbb{M}(T_\nu - S_\nu) \\ (3.2) \quad &\leq \liminf_{k \rightarrow \infty} [\mathbb{M}(S_\nu) + \mathbb{M}(T_\nu - S_\nu)] \\ &\leq \liminf_{k \rightarrow \infty} \mathbb{M}(T_\nu) = \mathbb{M}(T), \end{aligned}$$

i.e. $S \leq T$. Since we also have $\mathbb{M}(T) \leq \mathbb{M}(S) + \mathbb{M}(T - S)$, the inequalities in (3.2) actually are equalities. Also, since $\mathbb{M}(T - S) \leq \liminf_\nu \mathbb{M}(T_\nu - S_\nu)$ we obtain $\mathbb{M}(S) = \liminf_\nu \mathbb{M}(S_\nu)$. This is also true for every subsequence of S_ν , hence we have full convergence of the sequence $\mathbb{M}(S_\nu)$ to $\mathbb{M}(S)$ as $\nu \rightarrow \infty$. \square

We give now the definition of a cycle.

Definition 3.7. *We say that $C \in \mathcal{M}_k(E)$ is a cycle of $T \in \mathcal{M}_k(E)$, if $C \leq T$ and $\partial C = 0$. We say that T is acyclic, if $C = 0$ is the only cycle of T .*

It is easy now to prove the possibility to find such a cycle of every current T , that $T - C$ is acyclic. Of course, such a representation of a current as a sum of a cycle and an acyclic current is not unique, as can be seen, for instance, on the example of a current defined by a curve going from the south pole of S^2 to the north pole along some big semicircle, then back to the south pole along another big semicircle and finally back again to the north pole along a third big semicircle.

Proposition 3.8. *Every current T contains a cycle C such that $T - C$ is acyclic.*

Proof. Define

$$\xi(T) := \sup\{\mathbb{M}(C) : C \text{ is a cycle of } T\}.$$

Let C_0 be a cycle of $T_0 := T$ such that $\mathbb{M}(C_0) \geq \xi(T_0)/2$ and let $T_1 := T_0 - C_0$. Proceeding by induction we can define a sequence of currents C_ν such that C_ν is a cycle of T_ν with $\mathbb{M}(C_\nu) \geq \xi(T_\nu)/2$ and $T_{\nu+1} := T_\nu - C_\nu$.

Let C be any cycle of $T_{\nu+1} = T_\nu - C_\nu$. Putting together $C_\nu \leq T_\nu$ and $C \leq T_\nu - C_\nu$, with the use of the triangle inequality $\mathbb{M}(C + C_\nu) \leq \mathbb{M}(C) + \mathbb{M}(C_\nu)$ we obtain

$$\mathbb{M}(C + C_\nu) + \mathbb{M}(T_\nu - C_\nu - C) \leq \mathbb{M}(C) + \mathbb{M}(C_\nu) + \mathbb{M}(T_\nu - C_\nu - C) = \mathbb{M}(T_\nu),$$

hence $\tilde{C} := C_\nu + C$ is a cycle of T_ν and $\mathbb{M}(\tilde{C}) = \mathbb{M}(C) + \mathbb{M}(C_\nu)$ (i.e. $C \leq \tilde{C}$). This means that $\xi(T_\nu) \geq \mathbb{M}(\tilde{C}) = \mathbb{M}(C_\nu) + \mathbb{M}(C)$, hence

$$\mathbb{M}(C) \leq \xi(T_\nu) - \mathbb{M}(C_\nu) \leq \xi(T_\nu)/2,$$

and in particular $\xi(T_{\nu+1}) \leq \xi(T_\nu)/2$.

One has therefore that

$$\mathbb{M}(C_\nu) \leq \xi(T_\nu) \leq \frac{\xi(T_0)}{2^\nu},$$

so that $\sum_\nu C_\nu$ is convergent in mass and hence so is the sequence $\{T_\nu\}$, since $T_\nu = T - \sum_{k=0}^{\nu-1} C_k$. Letting $T' := \lim_\nu T_\nu$, we have $T' \leq T$ by Lemma 3.6 and we claim that T' is acyclic. In fact, let $\nu \in \mathbb{N}$ be arbitrary. Since $T_{\nu+k} \leq T_\nu$ (because, in fact, $T_{\nu+k} \leq \dots \leq T_{\nu+1} \leq T_\nu$) for all $k \in \mathbb{N}$, then passing to the limit as $k \rightarrow \infty$ we get again by Lemma 3.6 that $T' \leq T_\nu$. Thus if C' is a cycle of T' , it is also a cycle of T_ν , so that $\mathbb{M}(C') \leq \xi(T_\nu)$, and since $\xi(T_\nu) \rightarrow 0$ we obtain that $C' = 0$. \square

4. SMIRNOV DECOMPOSITION OF CURRENTS

To each $\theta \in \Theta$ we associate the integral one-dimensional current $[\![\theta]\!]$ defined by

$$[\![\theta]\!](f \, d\pi) := \int_0^1 f(\theta(t)) \, d\pi(\theta(t)) = \theta_{\#}[\![0, 1]\!](f \, d\pi)$$

(note that the latter integral does not depend on the parameterization of θ so it is well defined on equivalence classes $\theta \in \Theta$). We also define the *parametric length* of θ as

$$\ell(\theta) := \int_0^1 |\dot{\theta}(t)| \, dt.$$

Clearly, one has $\mathbb{M}([\![\theta]\!]) \leq \ell(\theta)$, while when θ is an arc, then

$$\mathcal{H}^1(\theta) = \mathbb{M}([\![\theta]\!]) = \ell(\theta).$$

The following rather simple assertion is valid.

Lemma 4.1. *If $\theta_\nu \in \Theta$ be curves with uniformly bounded length, $\ell(\theta_\nu) \leq C < +\infty$ for all $\nu \in \mathbb{N}$, and $\theta_\nu \rightarrow \theta \in \Theta$ as $\nu \rightarrow \infty$, then $[\![\theta_\nu]\!](f \, d\pi) \rightarrow [\![\theta]\!](f \, d\pi)$ for every $f \, d\pi \in D^1(E)$. In other words, the map $\theta \in \Theta \mapsto [\![\theta]\!]$ is a continuous embedding of each subset of curves from Θ with uniformly bounded lengths into the space of integral one-dimensional currents endowed with weak topology of currents.*

Proof. Note that

$$[\![\theta_\nu]\!](f \, d\pi) = [\![0, 1]\!](f \circ \theta_\nu \, d\pi \circ \theta_\nu),$$

so that the statement follows from the basic continuity property (theorem 3.5(ii) from [1]) of currents (alternatively, recalling $[\![\theta_\nu]\!](f \, d\pi) = \int_0^1 (f \circ \theta_\nu)(x) \, d(\pi \circ \theta_\nu)(x)$, one could have used just elementary calculus). \square

Further on we call any finite positive Borel measure η on Θ a *transport*, because it may be interpreted, roughly speaking, as the information on the amount of mass transported over each curve $\theta \in \Theta$. Given a transport η on Θ we define a functional T_η on $D^1(E)$ by the formula

$$(4.1) \quad T_\eta(\omega) := \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta(\theta)$$

for an $\omega \in D^1(E)$. The following theorem shows that T_η is a normal current under natural assumptions on η .

Theorem 4.2. *Let η be a transport satisfying*

$$\int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta) < +\infty.$$

Then (4.1) defines a normal one-dimensional current $T = T_\eta$ with

$$\partial T = \eta(1) - \eta(0), \text{ where } \eta(i) := (e_i)_\# \eta, \quad e_i(\theta) := \theta(i), \quad i = 0, 1.$$

In particular, if $\eta(1) \wedge \eta(0) = 0$, then

$$(\partial T)^+ = \eta(1), \quad (\partial T)^- = \eta(0),$$

where $(\partial T)^\pm$ are the positive and the negative part of the measure ∂T respectively. Furthermore, for all Borel sets $e \subset E$ one has

$$(4.2) \quad \mu_T(e) \leq \int_{\Theta} \mu_{\llbracket \theta \rrbracket}(e) d\eta(\theta).$$

Proof. It suffices to prove that $T = T_\eta$ has finite mass and finite boundary mass. According to the definition of mass

$$|T(f d\pi)| \leq \int_{\Theta} \left(\text{Lip}(\pi) \int_E |f| d\mu_{\llbracket \theta \rrbracket} \right) d\eta(\theta),$$

which gives (4.2) and hence in particular shows that T has finite mass

$$\mathbb{M}(T) \leq \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta) < +\infty.$$

Finally, the calculation

$$\begin{aligned} \partial T(f) &= T(1 df) = \int_{\Theta} \left(\int_0^1 df(\theta(t)) \right) d\eta(\theta) = \int_{\Theta} [f(\theta(1)) - f(\theta(0))] d\eta(\theta) \\ &= \int_{\Theta} f(e_1(\theta)) d\eta(\theta) - \int_{\Theta} f(e_0(\theta)) d\eta(\theta) = \int_E f(x) d(\eta(1) - \eta(0)) \end{aligned}$$

concludes the proof. \square

It is worth mentioning that the inequality in (4.2) may be strict, as the following example shows.

Example 4.3. Let \bar{e}_i , $i = 1, 2$ stand for the unit vectors along axis x_i in \mathbb{R}^2 , and let $\Theta_1 \subset \Theta$ be a set of paths θ in $Q := [0, 1] \times [0, 1]$ admitting a parameterization $\theta(t) = (t, x_2)$, $t \in [0, 1]$, for some $x_2 \in [0, 1]$. Define η_1 by the formula

$$\eta_1(e) := \mathcal{H}^1(e_0(e \cap \Theta_1))$$

for all Borel $e \subset \Theta$, where $e_0(\theta) := \theta(0)$. Clearly, $T_{\eta_1} = \bar{e}_1 \wedge \mathcal{L}^2 \llcorner Q$. Analogously, letting $\Theta_2 \subset \Theta$ be a set of paths θ admitting a parameterization $\theta(t) = (x_1, t)$, $t \in [0, 1]$, for some $x_1 \in [0, 1]$, and defining η_2 by the formula

$$\eta_2(e) := \mathcal{H}^1(e_0(e \cap \Theta_2))$$

for all Borel $e \subset \Theta$, we get $T_{\eta_2} = \bar{e}_2 \wedge \mathcal{L}^2 \llcorner Q$. Now, setting $\eta := \eta_1 + \eta_2$, one has $T_\eta = T_{\eta_1} + T_{\eta_2} = (\bar{e}_1 + \bar{e}_2) \wedge \mathcal{L}^2 \llcorner Q$, and hence, $\mathbb{M}(T_\eta) = \sqrt{2}$, while

$$\int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) d\eta = \int_{\Theta_1} \mathbb{M}(\llbracket \theta \rrbracket) d\eta_1 + \int_{\Theta_2} \mathbb{M}(\llbracket \theta \rrbracket) d\eta_2 = 2 > \mathbb{M}(T_\eta).$$

We now consider a converse statement, i.e. when for a given normal current $T \in \mathcal{M}_1(E)$, there is a transport η satisfying $T = T_\eta$. For this purpose we give the following definition.

Definition 4.4. *We say that a normal current $T \in \mathcal{M}_1(E)$ is decomposable in curves, if there is a transport η satisfying $T = T_\eta + C$, where $C \leq T$ is a cycle of T , $T_\eta \leq T$, and the equalities*

$$(4.3) \quad \mathbb{M}(T_\eta) = \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta),$$

and

$$(4.4) \quad \eta(1) = (\partial T)^+, \quad \eta(0) = (\partial T)^-$$

are valid.

Remark 4.5. In view of Theorem 4.2, the claim (4.3) is equivalent to a formally weaker one

$$\mathbb{M}(T) \geq \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta).$$

Note that the property of being decomposable in curves for acyclic currents (i.e. with $C = 0$) is exactly what is claimed in theorem C from [6] for classical Whitney one-dimensional normal acyclic currents in a Euclidean space ($E = \mathbb{R}^n$); if one decides to be meticulous, one has to mention also that there, instead of claim (4.4), a different (though equivalent) claim

$$\mu_{\partial T} = \int_{\Theta(E)} (\delta_{\theta(0)} + \delta_{\theta(1)}) d\eta(\theta)$$

is formulated. In Theorem 5.1 we will show this property for all one-dimensional Ambrosio-Kirchheim normal acyclic currents in any metric space (up to the set-theoretic assumption made in the Introduction), thus generalizing the mentioned result from [6] to metric currents.

We now are able to prove the following statement which is the principal tool of this paper.

Proposition 4.6. *Let $T \in \mathcal{M}_1(E)$ be an acyclic normal current such that there is a sequence of normal currents $T_\nu \in \mathcal{M}_1(E)$ decomposable in curves with $T_\nu \rightharpoonup T$ weakly in the sense of currents, $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ in the narrow sense of measures, and $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$. Then T is decomposable in curves, and in particular $T = T_\eta$ for some transport η , and η -a.e. $\theta \in \Theta$ is an arc.*

Remark 4.7. If $T \in \mathcal{M}_1(E)$ is an acyclic normal current decomposable in curves, then applying the above theorem for $T_\nu := T$ we get that in particular $T = T_\eta$ for some transport η such that relationships (4.3) and (4.4) hold, and η -a.e. $\theta \in \Theta$ is an arc.

Thus, a generic (not necessarily acyclic) normal current $T \in \mathcal{M}_1(E)$ is decomposable in curves, if and only if there is a transport η satisfying all the properties of Definition 4.4 (i.e. $T = T_\eta + C$, where $T_\eta \leq T$ is acyclic and $C \leq T$ is a cycle of T , the relationships (4.3) and (4.4) holding true), with the additional property that η -a.e. $\theta \in \Theta$ is an arc.

Remark 4.8. In the proof of the above Proposition 4.6 (in particular, in the key auxiliary instrument, Proposition 4.9) we heavily rely on the fact that the measures μ_{T_ν} as well as $(\partial T_\nu)^\pm$ are uniformly tight. For μ_{T_ν} this is true in view of Lemma B.2, while for $(\partial T_\nu)^\pm$ directly from the Prokhorov theorem for nonnegative measures (theorem 8.6.4 from [4]). However, in both arguments one silently admits the set-theoretic assumption made in the Introduction; without the latter one has to assume that each of the measures μ_{T_ν} , $(\partial T_\nu)^\pm$, μ_T and $(\partial T)^\pm$ is tight itself. Under such an assumption the statement remains true minding Remark B.3 (for uniform tightness of μ_{T_ν}) as well as theorem 8.6.4 from [4] (for uniform tightness of $(\partial T_\nu)^\pm$).

Proof. We may assume $\mathbb{M}(T_\nu) \leq C$ (one can take e.g. $C := \mathbb{M}(T) + 1$). Decomposability of T_ν in curves means the existence for each T_ν of a transport η_ν such that

$$\begin{aligned}
 T_\nu(f \, d\pi) &= T_{\eta_\nu}(f \, d\pi) + T'_\nu(f \, d\pi), & T'_\nu &\leq T_\nu, & \partial T'_\nu &= 0, \\
 T_{\eta_\nu}(f \, d\pi) &= \int_{\Theta} \llbracket \theta \rrbracket (f \, d\pi) \, d\eta_\nu(\theta), \\
 \mathbb{M}(T_{\eta_\nu}) &= \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) \, d\eta_\nu(\theta) = \int_{\Theta} \ell(\theta) \, d\eta_\nu(\theta), \\
 \eta_\nu(1) &= (\partial T_{\eta_\nu})^+, & \eta_\nu(0) &= (\partial T_{\eta_\nu})^-
 \end{aligned}
 \tag{4.5}$$

for all $f \, d\pi \in D^1(E)$ (in particular, by any of the last two equalities, the total masses $\eta_\nu(\Theta)$ are uniformly bounded). Since $\mathbb{M}(T'_\nu) \leq \mathbb{M}(T_\nu) \leq C$, $\partial T'_\nu = 0$, and $\mu_{T'_\nu} \leq \mu_{T_\nu}$ by Remark 3.5, the latter measures being uniformly tight, by Lemma B.2, hence so being also the former, then by compactness theorem 5.2 from [1] one has $T'_\nu \rightarrow T'$ with $T' \leq T$ and $\partial T' = 0$ by Lemma 3.6. Since T is acyclic, then $T' = 0$, and hence $T_{\eta_\nu} \rightarrow T$.

In view of (4.5) we have the estimate

$$\int_{\Theta} \ell(\theta) \, d\eta_\nu = \mathbb{M}(T_{\eta_\nu}) \leq C.$$

We may invoke therefore Proposition 4.9 below, obtaining that up to a subsequence (not relabeled) $\eta_\nu \rightarrow \eta$ in the narrow sense of measures for some finite Borel measure η , and, moreover, that one may pass to the limit as $\nu \rightarrow \infty$ in both sides of the first relationship of (4.5) obtaining therefore $T(f \, d\pi) = T_\eta(f \, d\pi)$ for each $f \, d\pi \in D^1(E)$, and hence $T = T_\eta$. One shows in addition that (4.4) is valid by passing to the limit as $\nu \rightarrow \infty$ in both sides of the last two equalities from (4.5).

Furthermore, note that

$$\mathbb{M}(T_{\eta_\nu}) = \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) \, d\eta_\nu(\theta) \tag{4.6}$$

by the second relationship of (4.5). Hence, minding that the functional $\theta \in \Theta \mapsto \mathbb{M}(\llbracket \theta \rrbracket)$ is l.s.c., and hence, the integral in the right-hand side of the above relationship is l.s.c. with respect to narrow convergence of η_ν , by passing to a limit in both sides of (4.6) as $\nu \rightarrow \infty$, we deduce

$$\mathbb{M}(T) = \lim_{\nu} \mathbb{M}(T_{\eta_\nu}) = \lim_{\nu} \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) \, d\eta_\nu(\theta) \geq \int_{\Theta} \mathbb{M}(\llbracket \theta \rrbracket) \, d\eta(\theta),$$

which provides (4.3) once one recalls Remark 4.5.

Consider also the functional defined over transports by

$$\eta \mapsto \int_{\Theta} \ell(\theta) \, d\eta.$$

It is l.s.c. with respect to the narrow convergence of measures (because the parametric length $\ell(\cdot)$ is l.s.c. in Θ). Hence, minding that for each η_ν one has

$$\mathbb{M}(T_\nu) = \int_{\Theta} \ell(\theta) d\eta_\nu = \mathbb{M}(T_\nu),$$

we get

$$\int_{\Theta} \ell(\theta) d\eta \leq \mathbb{M}(T).$$

Minding that the opposite inequality holds in view of Theorem 4.2, we get that in fact the above inequality is the equality and in particular, we have that $\mathbb{M}(\llbracket \theta \rrbracket) = \ell(\theta)$ for η -a.e. $\theta \in \Theta$.

Let $f: \Theta \rightarrow \Theta$ and $g: \Theta \rightarrow \Theta$ be given by Lemma 4.15. Then, minding

$$\mathbb{M}(\llbracket g(\theta) \rrbracket) \leq \ell(g(\theta)), \quad \mathbb{M}(\llbracket f(\theta) \rrbracket) \leq \ell(f(\theta)),$$

we get

$$\mathbb{M}(\llbracket g(\theta) \rrbracket) + \mathbb{M}(\llbracket f(\theta) \rrbracket) \leq \ell(g(\theta)) + \ell(f(\theta)) = \ell(\theta) = \mathbb{M}(\llbracket \theta \rrbracket),$$

hence $\mathbb{M}(\llbracket g(\theta) \rrbracket) + \mathbb{M}(\llbracket f(\theta) \rrbracket) = \mathbb{M}(\llbracket \theta \rrbracket)$ for η -a.e. $\theta \in \Theta$. By virtue of this one has that $T_{f\#\eta} \leq T$ and therefore $T_{f\#\eta}$ is a cycle of $T = T_\eta$. Hence, $T_{f\#\eta} = 0$, so that $T_{g\#\eta} = T_\eta = T$. This also means $\llbracket f(\theta) \rrbracket = 0$ for η -a.e. $\theta \in \Theta$. For such θ we have thus $\llbracket g(\theta) \rrbracket = \llbracket \theta \rrbracket$, and in particular, the chain of inequalities

$$\ell(\theta) = \mathbb{M}(\llbracket \theta \rrbracket) = \mathbb{M}(\llbracket g(\theta) \rrbracket) \leq \ell(g(\theta)) \leq \ell(\theta)$$

is true, which gives $\ell(g(\theta)) = \ell(\theta)$, hence η -a.e. $\theta \in \Theta$ is an arc as claimed. \square

The statement below is the key technical result used in the proof of Proposition 4.6.

Proposition 4.9. *Let $\{\eta_\nu\}$ be a sequence of nonnegative finite Borel measures over Θ with uniformly bounded total masses, and denote $T_\nu := T_{\eta_\nu}$. Assume that $T_\nu \rightharpoonup T$ weakly in the sense of currents, $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$, and*

$$(4.7) \quad \mathbb{M}(T_\nu) = \int_{\Theta} \ell(\theta) d\eta_\nu \leq C < +\infty$$

for all $\nu \in \mathbb{N}$, while the current T is acyclic. Then there exists a transport η such that up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ (and in particular, $\eta_\nu(i) \rightarrow \eta(i)$, $i = 0, 1$) in the narrow sense of measures, while $T = T_\eta$.

Remark 4.10. If in the statement of Proposition 4.9 we required that all η_ν be concentrated on some compact subset of $C \subset \Theta$ of curves with uniformly bounded lengths, then the assumption of acyclicity of the limit current T is unnecessary and the proof is quite immediate. In fact, in this case one has that, up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ as $\nu \rightarrow \infty$ in the $*$ -weak sense of measures over C for some finite Borel measure η over C . Then one immediately gets

$$T_\nu(\omega) = \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta_\nu(\theta) \rightarrow \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta(\theta) = T_\eta(\omega)$$

as $\nu \rightarrow \infty$, since the function $\theta \in C \mapsto \llbracket \theta \rrbracket(\omega)$ is continuous by Lemma 4.1. Hence $T = T_\eta$. The convergence $\eta_\nu(i) \rightarrow \eta(i)$, $i = 0, 1$, as $\nu \rightarrow \infty$ follows from the fact that a push-forward operator by means of a continuous function is continuous with respect to $*$ -weak convergence of measures.

Thus, the main difficulty in proving Proposition 4.9 is that we cannot say a priori that η_ν are concentrated in some compact subset of Θ . In this case acyclicity of the limit current will be important as we will see in Example 4.14 below.

Proof. Combine Corollary 4.12 and Lemma 4.13 below. \square

The results below are used in the proof of Proposition 4.9.

Lemma 4.11. *Under the conditions of Proposition 4.9 there is an increasing sequence of compact sets $\Theta'_k \subset \Theta$ such that $\eta_\nu((\Theta'_k)^c) \leq C/2^k$ for some $C > 0$ independent of k and ν and for all $\nu \in \mathbb{N}$. This is true without any assumption on the acyclicity of the limit current T .*

Proof. By uniform tightness of μ_{T_ν} there is a sequence of compact sets $K_k \subset E$ such that $\mu_{T_\nu}(K_k^c) \leq 1/4^k$. Consider the set

$$\Theta_k := \{\theta \in \Theta : \mu_{\llbracket \theta \rrbracket}(K_k^c) > 1/2^k\}.$$

One has then

$$\begin{aligned} \frac{1}{4^k} &\geq \mu_{T_\nu}(K_k^c) = \int_{\Theta} \mu_{\llbracket \theta \rrbracket}(K_k^c) d\eta_\nu(\theta) \\ &\geq \int_{\Theta_k} \mu_{\llbracket \theta \rrbracket}(K_k^c) d\eta_\nu(\theta) > \frac{1}{2^k} \eta_\nu(\Theta_k), \end{aligned}$$

so that $\eta_\nu(\Theta_k) \leq 1/2^k$. Letting then

$$\hat{\Theta}_j := \bigcap_{k \geq j} \Theta_k^c,$$

we get

$$(4.8) \quad \eta_\nu(\hat{\Theta}_j^c) = \eta_\nu\left(\bigcup_{k \geq j} \Theta_k\right) \leq \sum_{k=j}^{\infty} \eta_\nu(\Theta_k) \leq \frac{1}{2^{j-1}}.$$

We also observe that for

$$\Theta^j := \{\theta \in \Theta : \ell(\theta) > 2^j\}$$

one has

$$C \geq \mathbb{M}(T_\nu) = \int_{\Theta} \ell(\theta) d\eta_\nu \geq \int_{\Theta^j} \ell(\theta) d\eta_\nu > 2^j \eta_\nu(\Theta^j),$$

hence $\eta_\nu(\Theta^j) < C/2^j$. Finally, minding that the measures $\mu_{\partial T_\nu}$ are also uniformly tight, we get the existence, of a sequence of compact sets $\tilde{K}_k \subset E$ such that for each $j \in \mathbb{N}$ and for

$$\tilde{\Theta}_j := \{\theta \in \Theta : \theta(0) \in \tilde{K}_j \text{ or } \theta(1) \in \tilde{K}_j\}$$

one has $\eta_\nu(\tilde{\Theta}_j^c) \leq 1/2^j$ for all $\nu \in \mathbb{N}$.

Let then

$$\Theta'_j := (\hat{\Theta}_j \setminus \Theta^j) \cap \tilde{\Theta}_j \cap \{\theta \in \Theta : \mathbb{M}(\llbracket \theta \rrbracket) = \ell(\theta)\}.$$

Note that for η_ν -a.e. θ one has $\mathbb{M}(\llbracket \theta \rrbracket) = \ell(\theta)$ (this is true in view of (4.7) minding (4.2) and the inequality $\mathbb{M}(\llbracket \theta \rrbracket) \leq \ell(\theta)$). Recalling then (4.8), we arrive at the estimate

$$(4.9) \quad \eta_\nu((\Theta'_j)^c) = \eta_\nu((\hat{\Theta}_j \setminus \Theta^j)^c \cup \tilde{\Theta}_j^c) = \eta_\nu(\hat{\Theta}_j^c \cup \Theta^j \cup \tilde{\Theta}_j^c) \leq \frac{C}{2^j},$$

and in particular, all η_ν are concentrated on

$$\Theta' := \bigcup_j \Theta'_j.$$

Observe now that each Θ'_j is a precompact subset of Θ . In fact, every $\theta \in \Theta'_j$ with constant velocity over $[0, 1]$ satisfies $\text{Lip } \theta \leq 2^j$, while

$$\begin{aligned}\mu_{\llbracket \theta \rrbracket}(K_k^c) &\leq \int_{\theta^{-1}(K_k^c)} |\dot{\theta}(t)| dt = \ell(\theta) \mathcal{L}^1(\theta^{-1}(K_k^c)), \\ \mu_{\llbracket \theta \rrbracket}(K_k) &\leq \int_{\theta^{-1}(K_k)} |\dot{\theta}(t)| dt = \ell(\theta) \mathcal{L}^1(\theta^{-1}(K_k)), \\ \mu_{\llbracket \theta \rrbracket}(E) &= \int_0^1 |\dot{\theta}(t)| dt = \ell(\theta),\end{aligned}$$

hence by summing the above two inequalities and comparing them to the third equality, we have that in fact the equalities hold, and thus in particular,

$$\mathcal{L}^1(\theta^{-1}(K_k^c)) = \mu_{\llbracket \theta \rrbracket}(K_k^c) / \ell(\theta).$$

Thus, for a sequence $\theta_\nu \in \Theta'_j$ one has that either

- there is a subsequence (not relabeled) such that $\ell(\theta_\nu) \geq c > 0$, which implies

$$\mathcal{L}^1(\theta^{-1}(K_k^c)) \leq \frac{1}{2^k c},$$

for each $k \geq j$, while $\text{Lip } \theta_\nu \leq \ell(\theta_\nu) \leq 2^j$, so that this subsequence is compact in Θ by Proposition 2.1, or

- $\ell(\theta_\nu) \rightarrow 0$ as $\nu \rightarrow \infty$. In this case one has that either $\theta_\nu(0) \in \tilde{K}_j$ or $\theta_\nu(1) \in \tilde{K}_j$ for a subsequence of $\nu \in \mathbb{N}$. Since both cases are symmetric, we assume $\theta_\nu(0) \in \tilde{K}_j$. One has up to a subsequence (not relabeled), $\theta_\nu(0) \rightarrow x \in \tilde{K}_j$. Then for every $z_\nu \in \theta_\nu$ due to the estimate

$$d(z_\nu, \theta_\nu(0)) \leq \ell(\theta_\nu)$$

we get $z_\nu \rightarrow x$, so that θ_ν converges in Θ to a constant curve concentrated on $x \in E$.

This concludes the proof of the Lemma. \square

Corollary 4.12. *With the notation of Lemma 4.11 under the conditions of Proposition 4.9 all η_ν are concentrated over the set $\Theta' := \cup_j \bar{\Theta}'_j$. Denoting by $\bar{\Theta}'$ the closure of the latter in the space of continuous functions $C([0, 1]; E)$ (factorized by parameterization) with the uniform metric, we have that up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ and $\eta_\nu(i) \rightharpoonup \eta(i)$, $i = 0, 1$, as $\nu \rightarrow \infty$ in the narrow sense of measures for some finite Borel measure η over $\bar{\Theta}'$ concentrated over Θ' . This is again true without any assumption on the acyclicity of the limit current T .*

Proof. Clearly $\bar{\Theta}'$ defined in the statement being proven is a Polish space (since already Θ' is σ -compact, hence separable), and since the sequence η_ν is uniformly tight on $\bar{\Theta}'$ and has uniformly bounded mass, then, up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ as $\nu \rightarrow \infty$ in the narrow sense of measures for some finite Borel measure η over $\bar{\Theta}'$. Note that in view of Lemma 4.11 one has $\eta_\nu((\Theta'_j)^c) \leq C/2^j$ for all $\nu \in \mathbb{N}$ (here and below the complement is meant now with respect to $\bar{\Theta}'$), hence $\eta((\bar{\Theta}'_j)^c) \leq C/2^j$ and therefore η is concentrated on Θ' . The convergence $\eta_\nu(i) \rightharpoonup \eta(i)$, $i = 0, 1$, as $\nu \rightarrow \infty$ again follows from continuity with respect to narrow convergence of measures of the push-forward operator by means of a continuous function. \square

Lemma 4.13. *Under the conditions of Proposition 4.9, let η be a limit point in the narrow topology of η_ν pointed out in Corollary 4.12. Then $T = T_\eta$.*

Proof. We use the notation of Lemma 4.11 and Corollary 4.12. Let us show first that

$$(4.10) \quad \phi(k) := \limsup_{\nu} \int_{C_k^c} \ell(\theta) d\eta_{\nu}(\theta) \rightarrow 0 \text{ when } k \rightarrow \infty,$$

where $C_k := \{\theta \in \bar{\Theta}' : \ell(\theta) \leq k\}$. It is here that the assumption on the acyclicity of T enters in play. To prove (4.10) assume the contrary. Then there exists a $c > 0$ such that for a subsequence of η_{ν} (not relabeled) one has

$$\int_{\{\ell(\theta) > \nu\}} \ell(\theta) d\eta_{\nu}(\theta) \geq c.$$

Consider then $\eta'_{\nu} := \eta_{\nu} \llcorner \{\ell(\theta) > \nu\}$, and $S_{\nu} := T_{\eta'_{\nu}}$. By Lemma 4.16, each S_{ν} is a subcurrent of T_{ν} , and hence by Lemma 3.6 one gets that up to a subsequence (again not relabeled) $S_{\nu} \rightarrow S$ weakly in the sense of currents as $\nu \rightarrow \infty$, while S is a subcurrent of T and $\mathbb{M}(S) \geq c$. On the other hand, $\eta'_{\nu} \rightarrow 0$ (in fact, even the total masses of η'_{ν} converge to zero) because by (4.7) one has

$$\nu \eta'_{\nu}(\Theta) = \nu \int_{\{\ell(\theta) > \nu\}} d\eta_{\nu} \leq \int_{\{\ell(\theta) > \nu\}} \ell(\theta) d\eta_{\nu} \leq \int_{\Theta} \ell(\theta) d\eta_{\nu} \leq C < +\infty.$$

Thus

$$\partial S_{\nu} = \eta'_{\nu}(1) - \eta'_{\nu}(0) \rightarrow 0$$

weakly in the sense of measures as $\nu \rightarrow \infty$ (again, in fact also in mass although we do not really need it), hence $\partial S = 0$ and, by acyclicity of T , one gets $S = 0$, giving a contradiction. Hence, the claim (4.10) is proven.

Fix now an arbitrary $\omega := f d\pi \in D^1(E)$ with $f \in L^{\infty}$, denoting for the sake of brevity $|\omega| := \|f\|_{\infty} \text{Lip } \pi$. Mind that

$$|\llbracket \theta \rrbracket(\omega)| \leq |\omega|k$$

whenever $\theta \in C_k$. For each $k \in \mathbb{N}$ using the classical Tietze-Urysohn extension theorem we can find a bounded continuous function $f_k : \bar{\Theta}' \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} f_k(\theta) &= \llbracket \theta \rrbracket(\omega), & \text{if } \theta \in C_k, \\ |f_k(\theta)| &\leq |\omega|k & \text{for all } \theta \in \bar{\Theta}'. \end{aligned}$$

We have now

$$(4.11) \quad \int_{\bar{\Theta}'} f_k(\theta) d\eta_{\nu}(\theta) \rightarrow \int_{\bar{\Theta}'} f_k(\theta) d\eta(\theta)$$

as $\nu \rightarrow \infty$. On the other hand,

$$\int_{\bar{\Theta}'} f_k(\theta) d\eta_{\nu}(\theta) - \int_{\bar{\Theta}'} \llbracket \theta \rrbracket(\omega) d\eta_{\nu}(\theta) = \int_{C_k^c} (f_k(\theta) - \llbracket \theta \rrbracket(\omega)) d\eta_{\nu}(\theta),$$

while

$$\begin{aligned} \left| \int_{C_k^c} (f_k(\theta) - \llbracket \theta \rrbracket(\omega)) d\eta_{\nu}(\theta) \right| &\leq \int_{C_k^c} |f_k(\theta)| d\eta_{\nu}(\theta) + \int_{C_k^c} |\llbracket \theta \rrbracket(\omega)| d\eta_{\nu}(\theta) \\ (4.12) \quad &\leq \int_{C_k^c} |\omega|k d\eta_{\nu}(\theta) + \int_{C_k^c} |\omega|\ell(\theta) d\eta_{\nu}(\theta) \\ &\leq 2|\omega| \int_{C_k^c} \ell(\theta) d\eta_{\nu}(\theta) \leq 4|\omega|\phi(k) \end{aligned}$$

for all sufficiently large $\nu \in \mathbb{N}$. Analogously,

$$(4.13) \quad \int_{\bar{\Theta}'} f_k(\theta) d\eta(\theta) - \int_{\bar{\Theta}'} \llbracket \theta \rrbracket(\omega) d\eta(\theta) = \int_{C_k^c} (f_k(\theta) - \llbracket \theta \rrbracket(\omega)) d\eta(\theta),$$

with

$$\left| \int_{C_k^c} (f_k(\theta) - \llbracket \theta \rrbracket(\omega)) d\eta(\theta) \right| \leq 2|\omega|\phi(k),$$

where we used the fact that since the set C_k is closed in $\bar{\Theta}'$, then by (4.10) one has

$$\int_{C_k^c} \ell(\theta) d\eta(\theta) \leq \phi(k).$$

Thus, in view of (4.11) and (4.12) we get

$$\begin{aligned} \int_{\Theta} f_k(\theta) d\eta(\theta) - 4|\omega|\phi(k) &\leq \liminf_{\nu} \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta_{\nu}(\theta) \leq \limsup_{\nu} \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta_{\nu}(\theta) \\ &\leq \int_{\Theta} f_k(\theta) d\eta(\theta) + 4|\omega|\phi(k). \end{aligned}$$

Minding (4.13), we arrive at the estimate

$$\begin{aligned} \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta(\theta) - 6|\omega|\phi(k) &\leq \liminf_{\nu} \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta_{\nu}(\theta) \leq \limsup_{\nu} \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta_{\nu}(\theta) \\ &\leq \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta(\theta) + 6|\omega|\phi(k), \end{aligned}$$

and passing to the limit as $k \rightarrow \infty$, we get

$$\lim_{\nu} T_{\nu}(\omega) = \int_{\Theta} \llbracket \theta \rrbracket(\omega) d\eta(\theta) = T_{\eta}(\omega)$$

as $\nu \rightarrow \infty$, which allows us to conclude that $T = T_{\eta}$. \square

It is worth remarking that the requirement of acyclicity of the limit current T of the above Proposition 4.9 is essential as shown in the example below.

Example 4.14. Consider the sequence of curves in \mathbb{R}^2 admitting the parameterization $\theta_{\nu}(t) := (1 + t/\nu)(\cos(2\pi\nu t), \sin(2\pi\nu t))$, $t \in [0, 1]$, and define $\eta_{\nu} := \frac{1}{\nu}\delta_{\theta_{\nu}}$ be the transport concentrated on $\theta_{\nu} \in \Theta$ and having total mass $1/\nu$. Define also $\bar{\theta}(t) := (\cos(2\pi t), \sin(2\pi t))$ and let $\eta := \delta_{\bar{\theta}}$ be the transport concentrated on $\bar{\theta}$ with unit total mass. Clearly $\eta_{\nu} \rightarrow 0$ in the narrow sense of measures as $\nu \rightarrow \infty$ (in fact, $\eta_{\nu}(\Theta) = 1/\nu$). On the other hand, $T_{\eta_{\nu}} \rightarrow T_{\eta} \neq 0$ as $\nu \rightarrow \infty$. However, this is not in contradiction with the above Proposition 4.9 because clearly $\partial T_{\eta} = 0$, i.e. T_{η} is a cycle.

Another lemma used in the proof of Proposition 4.6 is provided below.

Lemma 4.15. *The following assertions are valid.*

- (i) *There is a map $f: \Theta \rightarrow \Theta$ measurable with respect to all transports such that $f(\theta)$ is a loop (i.e. a simple closed curve) contained in $\theta \in \Theta$ with*

$$\ell(f(\theta)) \geq 1/2 \sup \{ \ell(\sigma) : \sigma \text{ is a loop contained in } \theta \}.$$

- (ii) *There is a map $g: \Theta \rightarrow \Theta$ measurable with respect to all transports such that for all $\theta \in \Theta$ one has $\theta = g(\theta) \cup f(\theta)$ (as traces), $\llbracket \theta \rrbracket = \llbracket g(\theta) \rrbracket + \llbracket f(\theta) \rrbracket$,*

$$\ell(g(\theta)) < \ell(\theta),$$

unless θ is an arc, and, finally, $g(\theta) = \theta$, if and only if θ is an arc.

Proof. We construct a map $f: \Theta \rightarrow \Theta$ satisfying claim (i) as follows. For every $\theta \in \Theta$ and $x \in \theta$ we let $C(\theta, x)$ stand for the set of curves contained in θ starting and ending at x in the sense that

$$\begin{aligned} C(\theta, x) &= \left\{ \tilde{\theta} \in \Theta : \tilde{\theta}(t) = \theta((1-t)s_1 + ts_2) \right. \\ &\quad \left. \text{for some } 0 \leq s_1 \leq s_2 \leq 1, \theta(s_1) = \theta(s_2) = x \right\}. \end{aligned}$$

In case $x \notin \theta$ we define $C(\theta, x)$ to be a set consisting just of a single curve θ_x defined by $\theta_x(t) := x$ for all $t \in [0, 1]$, i.e. of a “constant” curve the trace of which reduces to just one point x . Note that $\theta_x \in C(\theta, x)$ for all $x \in E$. Defined in this way, the multivalued map

$$(\theta, x) \in \Theta \times \mathbb{R}^n \mapsto C(\theta, x) \subset \Theta$$

is u.s.c. (as a multivalued map), and hence Borel measurable. Therefore, recalling that $\ell: \Theta \rightarrow \mathbb{R}$ is l.s.c. one gets the Borel measurability of the single-valued map

$$\lambda: \theta \in \Theta \mapsto \sup_{x \in \mathbb{R}^n} \sup\{\ell(\sigma) : \sigma \in C(\theta, x)\} \in \mathbb{R}.$$

Clearly, $\lambda(\theta)$ gives is the supremum of the length of the loops contained in θ . Finally, we define

$$F: \theta \in \Theta \mapsto \left\{ \sigma \in \bigcup_{x \in \theta} C(\theta, x) : \ell(\sigma) \geq \lambda(\theta)/2 \right\} \subset \Theta.$$

By the von Neumann-Aumann measurable selection theorem [7][corollary 5.5.8] one can find a selection $f: \Theta \rightarrow \Theta$ of the multivalued map F which is measurable with respect to all transports η . Clearly, $f(\theta)$ is as announced in the statement being proven.

Define now $g: \Theta \rightarrow \Theta$ as a union of two curvilinear segments, by setting

$$g(\theta) := [\theta(0), f(\theta)(0)] \circ [f(\theta)(1), \theta(1)].$$

Clearly, $g(\theta)$ is obtained by “cancelling” the loop $f(\theta)$ from θ . The properties of g announced in claim (ii) follow immediately since $\ell(g(\theta)) \leq \ell(\theta) - \lambda(\theta)/2$, while $g(\theta) = \theta$, if and only if $f(\theta) = \theta_x$ for some $x \in \theta$, i.e. when θ is an arc. \square

Finally, the following two elementary observations have also to be mentioned because used in the proof, as well as because of having some independent interest.

Lemma 4.16. *Let η be a transport satisfying (4.3) for a normal current $T = T_\eta \in \mathcal{M}_1(E)$ with compact support, and let $\tilde{\eta}$ be another transport such that $\tilde{\eta} \leq \eta$. Then for $\tilde{T} := T_{\tilde{\eta}}$ one has $\tilde{T} \leq T$ and $\mathbb{M}(\tilde{T}) = \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\tilde{\eta}(\theta)$.*

Proof. Let $\eta' := \eta - \tilde{\eta}$ and $T' := T_{\eta'}$. By Theorem 4.2 one has

$$\mathbb{M}(\tilde{T}) \leq \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\tilde{\eta}(\theta), \quad \mathbb{M}(T') \leq \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\eta'(\theta),$$

and therefore, minding that $T = \tilde{T} + T'$, we get

$$\begin{aligned} \mathbb{M}(T) &\leq \mathbb{M}(\tilde{T}) + \mathbb{M}(T') \leq \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\tilde{\eta}(\theta) + \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\eta'(\theta) \\ &= \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta) = \mathbb{M}(T), \end{aligned}$$

which implies that all the above inequalities are actually equalities and hence the thesis follows. \square

Lemma 4.17. *Let $T \in \mathcal{M}_1(E)$ be a normal current and η be such a transport that $T = T_\eta$ and $\mathbb{M}(T) = \int_\Theta \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta)$. Then $\mu_T = \mu_{\llbracket \theta \rrbracket} \otimes \eta$, i.e.*

$$(4.14) \quad \mu_T(e) = \int_\Theta \mu_{\llbracket \theta \rrbracket}(e) d\eta(\theta),$$

and, moreover,

$$(4.15) \quad T \llcorner \phi(f d\pi) = \int_\Theta \llbracket \theta \rrbracket \llcorner \phi(f d\pi) d\eta(\theta),$$

for every Borel function $\phi: E \rightarrow \mathbb{R}$ and every $f d\pi \in D^1(E)$.

Proof. By Theorem 4.2 one has

$$\mu_T(e) \leq \int_{\Theta} \mu_{\llbracket \theta \rrbracket}(e) d\eta(\theta)$$

for every Borel set $e \subset E$, while according to the assumptions the latter estimate becomes an equality for $e := E$. Thus (4.14) follows. The relationship (4.15) is just an easy calculation

$$\begin{aligned} T \llcorner \phi(f d\pi) &= T(f \phi d\pi) = \int_{\Theta} \llbracket \theta \rrbracket(f \phi d\pi) d\eta(\theta) \\ &= \int_{\Theta} \llbracket \theta \rrbracket \llcorner \phi(f d\pi) d\eta(\theta), \end{aligned}$$

which therefore concludes the proof. \square

5. CURRENTS DECOMPOSABLE IN CURVES

Our aim is now to prove the following theorem, which is the main result of the paper.

Theorem 5.1. *Let E be a complete metric space. Then every acyclic normal one-dimensional real current T in E is decomposable in curves, so that in particular, there is a transport η satisfying $T = T_\eta$, such that relationships (4.3) and (4.4) hold, and η -a.e. $\theta \in \Theta$ is an arc.*

Remark 5.2. Let us emphasize that the statement of the above theorem is true in every complete metric space, since we assumed in the Introduction that the density character of every metric space is an Ulam number. Without such an assumption this result still holds in view of Remark 5.6 when E is an arbitrary complete metric space and μ_T and $\mu_{\partial T}$ are tight measures, hence in particular for every $T \in \mathcal{M}_1(E)$ once E is a Polish (i.e. complete separable) metric space.

To this aim we first provide several technical statements.

First we prove a similar decomposition statement for one-dimensional real *polyhedral* currents in a finite-dimensional normed space.

Lemma 5.3. *Let E be a finite-dimensional normed space and $T \in \mathcal{M}_1(E)$ be an acyclic polyhedral current over E , i.e. $T = \sum_{\nu=1}^N \theta_\nu T_\nu$, where $\theta_\nu > 0$, and $T_\nu = \llbracket a_\nu, b_\nu \rrbracket$ are currents associated to oriented segments which may overlap only at endpoints. Then there exists a Borel measure η over Θ such that $T = T_\eta$ and relationships (4.3) and (4.4) hold, while η -a.e. $\theta \in \Theta$ is an arc.*

Proof. Let us call *edges* the oriented segments $T_\nu = \llbracket a_\nu, b_\nu \rrbracket$, $\nu = 1, \dots, N$. We say that an ordered finite collection of edges $(T_{\nu_1}, \dots, T_{\nu_M})$, where $T_{\nu_i} := \llbracket a_{\nu_i}, b_{\nu_i} \rrbracket$, $i = 1, \dots, M$, is a *path* in T , if $b_{\nu_i} = a_{\nu_{i+1}}$ for $i = 1, \dots, M-1$. We say that such a path is *closed*, if also $b_{\nu_M} = a_{\nu_1}$. Clearly an acyclic T contains no closed paths. Given a path in T , we can extend it *forward*, if there exists an edge T_ν of T such that $a_\nu = b_{\nu_M}$, and *backward*, if there exists an edge T_ν such that $b_\nu = a_{\nu_1}$.

Let $\bar{\nu}$ be such that $\theta_{\bar{\nu}} = \min\{\theta_1, \dots, \theta_N\}$ and consider the path $(T_{\bar{\nu}})$ with a single edge. Then extend this path as much as possible forward and backward. At each extension step the path cannot become closed, hence the path is composed by all different edges. Since there is only a finite number of edges in T , this extension process must finish in a finite number of steps. We obtain in this way a *maximal path* containing $T_{\bar{\nu}}$. Let $(T_{\nu_1}, \dots, T_{\nu_M})$ be this maximal path and consider the corresponding current

$$P_0 := \theta_{\bar{\nu}} \sum_{i=1}^M T_{\nu_i}.$$

Clearly, $P_0 \leq T$ and $\partial P_0 = \llbracket b_{\nu_M} \rrbracket - \llbracket a_{\nu_1} \rrbracket$. Since the path is maximal, in T there is no edge $T_\nu = \llbracket a_\nu, b_\nu \rrbracket$ with endpoint $b_\nu = a_{\nu_1}$, and thus $(\partial P_0)^- = \theta_\nu \llbracket a_{\nu_1} \rrbracket$ is a subcurrent of $(\partial T)^-$. Analogously $(\partial P_0)^+ = \theta_\nu \llbracket b_{\nu_M} \rrbracket \leq (\partial T)^+$.

To represent P_0 as a measure on Θ we just consider the curve σ_0 representing the polygonal path $[a_{\nu_1}, b_{\nu_1}] \circ \dots \circ [a_{\nu_M}, b_{\nu_M}]$ and the Dirac measure $\eta_0 := \theta_\nu \delta_{\sigma_0}$ to obtain $P_0 = T_{\eta_0}$. Clearly $\eta_0(1) = (\partial P_0)^+$ and $\eta_0(0) = (\partial P_0)^-$.

The current $T' := T - P_0$ is itself a polyhedral acyclic current with $\partial T' \leq \partial T$ (since $\partial P_0 \leq \partial T$ as noted above). Moreover T' can be represented with strictly less edges than T because the edge T_ν has been removed from T . Hence repeating the previous construction with T' in place of T we find a subcurrent P_1 representing a path in T' and such that $P_1 = T_{\eta_1}$ with $\eta_1(1) = (\partial P_1)^+ \leq (\partial T)^+$ and $\eta_1(0) = (\partial P_1)^- \leq (\partial T)^-$. A finite number of such steps will eventually exhaust T and yield a decomposition $T = \sum_{i=0}^k P_i$ such that the corresponding measure $\eta := \sum_{i=0}^k \eta_i$ has the required properties. \square

Lemma 5.4. *Let E be a finite-dimensional normed space. Then every acyclic normal current $T \in \mathcal{M}_1(E)$ with bounded support in E is decomposable in curves, so that in particular, there is a transport η satisfying $T = T_\eta$, while relationships (4.3) and (4.4) hold, and η -a.e. $\theta \in \Theta$ is an arc.*

Proof. Combine Lemmata 5.3 and C.1 with Proposition 4.6. \square

Lemma 5.5. *Let E be a Banach space with metric approximation property. Then every acyclic normal current $T \in \mathcal{M}_1(E)$ in E is decomposable in curves, so that in particular, there is a transport η satisfying $T = T_\eta$, while relationships (4.3) and (4.4) hold, and η -a.e. $\theta \in \Theta$ is an arc.*

Remark 5.6. The Lemma 5.5 is proven under the set-theoretic assumption made in the Introduction. Without this assumption one has to assume that the measures μ_T and $\mu_{\partial T}$ are tight. Then the statement of the Lemma is still true with the following argument added to the proof. In fact, in the notation of the proof, one has $P_{n\#}\mu_T \rightharpoonup \mu_T$ in the narrow sense of measures when $n \rightarrow \infty$, while the measures $P_{n\#}\mu_T$ are tight (in fact, they are concentrated over the σ -compact set $P_n(\cup_\nu K_\nu) = \cup_\nu P_n(K_\nu)$), so that in particular, the measures $P_{n\#}\mu_T$ are uniformly tight by theorem 8.6.4 from [4]. But, minding $\|P_n\| \leq 1$, we have $\mu_{T_n} \leq P_{n\#}\mu_T$, which means that the measures μ_{T_n} are also uniformly tight. Analogously, we have that the measures $\mu_{\partial T_n} = (\partial T_n)^+ + (\partial T_n)^-$ are also uniformly tight, and hence so are the measures $(\partial T_n)^\pm$. The Proposition 4.6 in the proof may then be invoked minding Remark 4.8.

Proof. Let $\{K_\nu\}$ be an increasing sequence of compact subsets of E such that μ_T and $\mu_{\partial T}$ are concentrated on $\cup_\nu K_\nu$, and let P_ν be a finite rank projection of norm one such that $\|P_\nu x - x\| \leq 1/\nu$ for all $x \in K_\nu$. Thus $P_\nu x \rightarrow x$ as $\nu \rightarrow \infty$ for all $x \in \cup_\nu K_\nu$.

Consider first the case when $\text{supp } T$ is bounded. Let $T_n := P_{n\#}T$. Clearly, $T_n \rightharpoonup T$ in the weak sense of currents. In fact, for every $f d\pi \in D^1(E)$ we have

$$\begin{aligned} |T(f \circ P_n d\pi \circ P_n) - T(f d\pi)| &\leq |T(f \circ P_n d\pi \circ P_n) - T(f \circ P_n d\pi)| + \\ &\quad |T(f \circ P_n d\pi) - T(f d\pi)| \\ &\leq \int_E |f \circ P_n| \cdot |\pi \circ P_n - \pi| d\mu_{\partial T} + \text{Lip } f \int_E |\pi \circ P_n - \pi| d\mu_T + \\ &\quad |T(f \circ P_n d\pi) - T(f d\pi)| \quad \text{by proposition 5.1 of [1]} \\ &\leq (\|f\|_\infty \text{Lip } \pi + \text{Lip } f \text{Lip } \pi) \int_E \|P_n x - x\| d(\mu_{\partial T} + \mu_T) + \\ &\quad |T(f \circ P_n d\pi) - T(f d\pi)|, \end{aligned}$$

all the terms in the right-hand side tending to zero as $n \rightarrow \infty$ by the choice of P_n (the first one by Lebesgue theorem, recalling that $\|P_n x - x\| \leq 2\|x\|$ and the support of T , and hence of ∂T , is bounded, while the last term because $f(P_n(x)) \rightarrow f(x)$ for μ_T -a.e. $x \in E$). Further, we have

$$\mathbb{M}(T) \leq \liminf_n \mathbb{M}(T_n) \leq \limsup_n \mathbb{M}(T_n) \leq \mathbb{M}(T),$$

since $\mathbb{M}(T_n) \leq \mathbb{M}(T)$, and therefore $\mathbb{M}(T_n) \rightarrow \mathbb{M}(T)$ as $n \rightarrow \infty$. Finally,

$$(\partial T_n)^\pm = P_{n\#}(\partial T)^\pm - P_{n\#}(\partial T)^+ \wedge P_{n\#}(\partial T)^-,$$

and thus minding that $P_{n\#}(\partial T)^\pm \rightharpoonup (\partial T)^\pm$, we get $(\partial T_n)^\pm \rightharpoonup (\partial T)^\pm$ as $n \rightarrow \infty$ in the narrow sense of measures. It suffices then to recall that T_n are decomposable in curves (as currents over a finite-dimensional space by Lemma 5.4) and apply Proposition 4.6.

For the general case of a current T with possibly unbounded support, we approximate T by a sequence $\{T_\nu\} \subset \mathcal{M}_1(E)$, such that each T_ν has bounded support and $\mathbb{M}(T_\nu - T) + \mathbb{M}(\partial T_\nu - \partial T) \rightarrow 0$ as $\nu \rightarrow \infty$ (for this purpose just take $T_\nu := T \llcorner g_\nu$ for a $g_\nu \in \text{Lip}_1(E)$ with bounded support having $0 \leq g_\nu \leq 1$ and $g_\nu = 1$ on $B_\nu(0)$). Now T_ν is decomposable in curves as just proven, while the whole sequence $\{T_\nu\}$ satisfies all the conditions of Proposition 4.6 (the only thing to verify is $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ in the narrow sense of measures, which is true in view of the corollary 8.4.8 from [4]), and invoking the latter we conclude the proof. \square

The following lemma is probably a folkloric fact which is however not easily found in the literature.

Lemma 5.7. ℓ^∞ has the metric approximation property.

Proof. One has to show the existence for every $\varepsilon > 0$ and every finite set $X \subset \ell^\infty$ of a finite-rank projection T with $\|T\| \leq 1$ such that $\|Tx - x\| < \varepsilon$ for all $x \in X$. In fact, then for every compact $K \subset \ell^\infty$ choosing a finite ε -net $X \subset K$, we get for all $x \in K$, choosing $y \in X$ so that $\|x - y\| \leq \varepsilon$, the estimate

$$\|Tx - x\| \leq \|Tx - Ty\| + \|Ty - y\| + \|y - x\| \leq 2\|x - y\| + \|Ty - y\| \leq 3\varepsilon.$$

We now construct a net of finite rank projections of norm one as follows. Let Λ be the directed set of all finite partitions of \mathbb{N} ordered by refinement. For every partition $P \in \Lambda$, $P = \{N_i\}_{i=1}^k$, $N_i \subset \mathbb{N}$ and all N_i pairwise disjoint, we define the finite rank projection $T = T_P$ by setting $(Tx)_j := x_{i_1}$ for all $j \in N_i$, where i_1 stands for the first (i.e. lowest) index in N_i . Clearly, for every $x \in \ell^\infty$ and every $\varepsilon > 0$ there is a partition $P_{x,\varepsilon} \in \Lambda$ such that $\|T_P x - x\| < \varepsilon$ for all $P \in \Lambda$ with $P > P_{x,\varepsilon}$ (such a partition is done by dividing the interval $[\inf x, \sup x]$ in subintervals I_i of length not exceeding ε , and taking $x^{-1}(I_i)$ to be the elements of P). Thus for a finite $X \subset \ell^\infty$ there is a partition $P_{X,\varepsilon} \in \Lambda$ such that $\|T_P x - x\| < \varepsilon$ for all $x \in X$ and for all $P \in \Lambda$ with $P > P_{X,\varepsilon}$ (just take $P_{X,\varepsilon} > P_{x,\varepsilon}$ for all $x \in X$). \square

Now we are able to prove Theorem 5.1.

Proof of Theorem 5.1. Note that under the set-theoretic assumption made in the Introduction μ_T is concentrated over $\text{supp } \mu_T =: \text{supp } T$, and the value of $T(f d\pi)$ for $f d\pi \in D^1(E)$ is completely determined by the restriction of f and π to $\text{supp } T$. In fact, if $f d\pi \in D^1(E)$, then $T(f d\pi) = T(f \cdot 1_{\text{supp } T} d\pi)$, and if $\pi = 0$ over $\text{supp } T$, then

$$T(f d\pi) = T(f \cdot 1_{\text{supp } T} d\pi) = 0,$$

so that if $f^i d\pi^i \in D^1(E)$, $i = 1, 2$, with $f^1 d\pi^1 \llcorner \text{supp } T = f^2 d\pi^2 \llcorner \text{supp } T$, then $T(f^1 d\pi^1) = T(f^2 d\pi^2)$.

Recalling that under the same set-theoretic assumption made in the Introduction the set $\text{supp } T \subset E$ is separable, we may just assume $E := \text{supp } T$ thus reducing to

the case of a complete separable metric space E . Denote by $j: E \rightarrow \ell^\infty$ an isometric embedding of E into ℓ^∞ . Combining Lemma 5.5 with Lemma 5.7 we get that $j_\#T$ is decomposable in curves, i.e. for some transport η' over $\Theta(\ell^\infty)$ one has

$$\begin{aligned} j_\#T(f' d\pi') &= \int_{\Theta(\ell^\infty)} \llbracket \theta' \rrbracket (f' d\pi') d\eta'(\theta'), \\ \mathbb{M}(j_\#T) &= \int_{\Theta(\ell^\infty)} \mathbb{M}(\llbracket \theta' \rrbracket) d\eta'(\theta'), \text{ and} \\ \eta'(1) &= (\partial j_\#T)^+, \quad \eta'(0) = (\partial j_\#T)^-, \end{aligned}$$

for all $f d\pi \in D^1(\ell^\infty)$, while η' -a.e. $\theta' \in \Theta(\ell^\infty)$ is an arc.

Note that j induces the isometric imbedding $\iota: \Theta(E) \rightarrow \Theta(\ell^\infty)$ by the formula

$$\iota(\theta)(t) := j(\theta(t))$$

for all $\theta \in \Theta(E)$ and $t \in [0, 1]$. Let $\Sigma \subset j(E) \subset \ell^\infty$ be a set such that $\mu_{j_\#T}(\Sigma^c) = j_\#\mu_T(\Sigma^c) = 0$. Then by Lemma 4.17 for η' -a.e. $\theta' \in \Theta(\ell^\infty)$ one has that $\mu_{\llbracket \theta' \rrbracket}$ is concentrated over Σ , hence $\theta'(s) \in \Sigma$ for a.e. $s \in [0, 1]$. Let $\theta(s) := j^{-1}(\theta'(s))$ for such s , and extend θ to the whole $[0, 1]$ by continuity, so that $\theta \in \Theta(E)$, and in particular, $\theta' = \iota(\theta)$. Thus one has that η' is concentrated over $\iota(\Theta(E))$, and hence we may define $\eta := \iota_\#^{-1}\eta'$. Note also that since η' -a.e. $\theta' \in \Theta(\ell^\infty)$ is an arc, then so is η -a.e. $\theta = \iota^{-1}(\theta') \in \Theta(E)$.

For $f d\pi \in D^1(E)$ we define $f' d\pi' \in D^1(\ell^\infty)$ by setting $f'(x) := f(j^{-1}(x))$, $\pi'(x) := \pi(j^{-1}(x))$ for $x \in j(E)$ and extending these functions to the whole ℓ^∞ . Then

$$j_\#T(f' d\pi') = T(f d\pi) \text{ and } \llbracket \theta' \rrbracket (f' d\pi') = j_\#\llbracket \theta \rrbracket (f' d\pi') = \llbracket \theta \rrbracket (f d\pi).$$

Hence, minding that $\eta' = \iota_\#\eta$, we get

$$T(f d\pi) = \int_{\Theta(E)} \llbracket \theta \rrbracket (f d\pi) d\eta(\theta).$$

Further, since $\mathbb{M}(T) = \mathbb{M}(j_\#T)$ and $\mathbb{M}(\llbracket \theta' \rrbracket) = \mathbb{M}(\llbracket \theta \rrbracket)$, one has

$$\mathbb{M}(T) = \int_{\Theta(E)} \mathbb{M}(\llbracket \theta \rrbracket) d\eta(\theta).$$

At last,

$$\eta(1) = (\iota_\#^{-1}\eta')(1) = j_\#^{-1}(\eta'(1)) = j_\#^{-1}(\partial j_\#T)^+ = (\partial T)^+,$$

and analogously,

$$\eta(0) = (\partial j_\#T)^-,$$

which concludes the proof. \square

APPENDIX A. AN APPLICATION TO OPTIMAL MASS TRANSPORTATION

In this section we provide an easy application of the representation result for acyclic metric currents to optimal mass transportation problems in metric space. It is not our goal to present such applications in full generality, but rather to illustrate the utility of the results proven in this paper.

Given two finite positive Borel measures φ^+ and φ^- of equal total mass with bounded (but not necessarily compact) support in a given metric space (E, d) , the classical Monge-Kantorovich optimal mass transportation problem in a metric space (E, d) is that of finding

$$(1.1) \quad \inf \left\{ \int_{E \times E} d(x, y) d\gamma(x, y) : \gamma \text{ admissible transport plan for } \varphi^+ \text{ and } \varphi^- \right\},$$

where by saying that γ is admissible, we mean that γ is a finite positive Borel measure over $E \times E$ satisfying the conditions on marginals

$$\pi_{\#}^{\pm} \gamma = \varphi^{\pm},$$

where $\pi^{\pm}: (x^{+}, x^{-}) \in E \times E \mapsto x^{\pm} \in E$. Recall that we are always assuming in this paper that finite positive Borel measures are tight (otherwise we just impose the tightness condition on φ^{+} and φ^{-}). The above infimum is clearly attained under such conditions. In fact in a minimizing sequence $\{\gamma_{\nu}\}$ of admissible transport plans, all plans have the same total masses (equal to the total mass of φ^{+} and φ^{-}) and the sequence is uniformly tight, because

$$\gamma_{\nu}((K \times K)^c) \leq \gamma(K^c \times E) + \gamma(E \times K^c) \leq 2\varepsilon$$

whenever $K \subset E$ is a compact set such that $\varphi^{\pm}(K^c) \leq \varepsilon$. Hence, by Prokhorov theorem for nonnegative measures (theorem 8.6.4 from [4]) γ_{ν} admits a narrow convergent subsequence, and therefore the existence of a minimizer follows from lower semicontinuity with respect to such a convergence of integrals with nonnegative lower semicontinuous integrands (in our case the integrand is even continuous). The value of the above infimum is usually denoted by $W_1(\varphi^{+}, \varphi^{-})$ and is called Wasserstein distance between φ^{+} and φ^{-} (or Kantorovich-Rubinstein distance, which should be surely more correct for historical reasons). Of course, to guarantee that $W_1(\varphi^{+}, \varphi^{-}) < +\infty$, extra conditions are required (usually one imposes conditions on the moments of φ^{\pm}).

The following result then holds true.

Theorem A.1. *Assume that E is a geodesic metric space (i.e. such that for every $(x, y) \in E \times E$ there is a curve $\theta \in \Theta$ connecting x to y such that $d(x, y) = \ell(\theta)$), and, moreover, there is a Borel map $q: \text{supp } \varphi^{+} \times \text{supp } \varphi^{-} \rightarrow \Theta(E)$ such that $d(x, y) = \ell(q(x, y))$. Then*

$$(1.2) \quad W_1(\varphi^{+}, \varphi^{-}) = \min\{\mathbb{M}(T) : T \in \mathcal{M}_1(E), \partial T = \varphi^{+} - \varphi^{-}\}.$$

Moreover, if T is a minimizer of the problem (1.2), then T is acyclic, and if η is a transport such that $T = T_{\eta}$ for which conditions of Theorem 5.1 hold, then $\gamma := (e_0 \times e_1)_{\#} \eta$ is a minimizer of (1.1), where $e_i(\theta) := \theta(i)$, $i = 0, 1$ for all $\theta \in \Theta(E)$.

Viceversa, when γ is a minimizer of (1.1), then setting $\eta := q_{\#} \gamma$ (so that in particular η is concentrated on a set of geodesics), we get that $T = T_{\eta}$ satisfies conditions of Theorem 5.1 and minimizes (1.2).

Remark A.2. The conditions of the above theorem are satisfied, for instance, in the following cases.

- (i) When E is a separable geodesic space. In fact, a map q indicated in the conditions exists in view of the Kuratowski-Ryll-Nardzewski measurable selection theorem 5.2.1 from [7] because the set

$$\{(\theta(0), \theta(1), \theta) \subset \text{supp } \varphi^{+} \times \text{supp } \varphi^{-} \times C([0, 1]; E) : d(\theta(0), \theta(1)) = \ell(\theta)\}$$

is closed (here the space $C([0, 1]; E)$ is assumed to be equipped with the usual uniform metric factorized by reparameterization of curves; further, it is assumed that $\ell(\theta) := +\infty$ for $\theta \in C([0, 1]; E)$ not rectifiable).

- (ii) When E is a Banach space (not necessarily separable). One may set then $q(x, y) := [x, y]$, where the curve $[x, y]$ is defined by

$$[x, y](t) := (1 - t)x + ty, \quad t \in [0, 1].$$

Remark A.3. The above result is clearly false in generic metric spaces. In particular, if one takes $E := [0, 1]$ equipped with the distance $d(x, y) := \sqrt{|x - y|}$, and $\varphi^{+} :=$

δ_0 , $\varphi^- := \delta_1$, then by Theorem 5.1 there is no current $T \in \mathcal{M}_1(E)$ such that $\partial T = \varphi^+ - \varphi^-$ (because $\Theta(E)$ reduces to only constant curves), so

$$\inf\{\mathbb{M}(T) : T \in \mathcal{M}_1(E), \partial T = \varphi^+ - \varphi^-\} = \inf \emptyset = +\infty,$$

while $W_1(\varphi^+, \varphi^-) = 1$ in this case. One has of course the same phenomenon if E is just the discrete space $E := \{0, 1\}$ with $d(0, 1) \neq 0$ and with the same choice of φ^\pm . This shows that in fact the minimization problem

$$\inf\{\mathbb{M}(T) : T \in \mathcal{M}_1(E), \partial T = \varphi^+ - \varphi^-\}$$

corresponds better to the idea of mass transportation than the classical Monge-Kantorovich setting.

Proof. Assume first $S \in \mathcal{M}_1(E)$ be such that $\partial S = \varphi^+ - \varphi^-$ and decompose $S = T + C$ with $C \leq S$, $\partial C = 0$, and $T \leq S$ acyclic by Proposition 3.8. If η is a transport such that $T = T_\eta$ for which conditions of Theorem 5.1 hold, then setting $\gamma := (e_0 \times e_1)_\# \eta$, we have that γ is admissible and

$$\begin{aligned} \mathbb{M}(S) &\geq \mathbb{M}(T) = \int_{\Theta(E)} \ell(\theta) d\eta(\theta) = \int_{\Theta(E)} d(\theta(0), \theta(1)) d\eta(\theta) \\ (1.3) \quad &= \int_{E \times E} d(x, y) d\gamma(x, y) \geq W_1(\varphi^+, \varphi^-). \end{aligned}$$

Further, let γ be a minimizer of (1.1), $\eta := q_\# \gamma$. We get then for $T = T_\eta$

$$\begin{aligned} W_1(\varphi^+, \varphi^-) &= \int_{E \times E} d(x, y) d\gamma(x, y) = \int_{E \times E} \ell(q(x, y)) d\gamma(x, y) \\ (1.4) \quad &= \int_{\Theta(E)} \ell(\theta) d\eta(\theta) \geq \mathbb{M}(T_\eta), \end{aligned}$$

the latter inequality being due to Theorem 4.2. Combined with (1.3) this gives the optimality of T for (1.2). In particular, equality holds in (1.4), so that η satisfies conditions of Theorem 5.1.

Finally, it remains to observe that every minimizer T of (1.2) is acyclic since deleting cycles decreases the mass without changing the boundary of a current. \square

Theorem A.1 shows the equivalence of three different descriptions of optimal mass transportation: the classical one in terms of transport plans γ proposed by Kantorovich which gives only the information on “who goes where” (i.e. only starting points and endpoints of transport paths), the one in terms of transports η (which is the most precise one since it gives the full information on paths covered by infinitesimal masses during transportation), and the intermediate one in terms of currents T giving the information on the total flow of mass. Of course, the respective claims can be obtained also without using representation Theorem 5.1 for acyclic currents. For instance the inequality

$$(1.5) \quad W_1(\varphi^+, \varphi^-) \leq \mathbb{M}(T)$$

for every $T \in \mathcal{M}_1(E)$ satisfying $\partial T = \varphi^+ - \varphi^-$ may be seen as a consequence of Kantorovich duality

$$W_1(\varphi^+, \varphi^-) = \sup \left\{ \int_E f d(\varphi^+ - \varphi^-) : f \in \text{Lip}_1(E) \right\}$$

coupled with the obvious relationship

$$\int_E f d(\varphi^+ - \varphi^-) = \partial T(f) = T(df) \leq \mathbb{M}(T)$$

whenever $f \in \text{Lip}_1(E)$. Together with (1.4) which is proven without use of the representation Theorem 5.1 (see the proof of Theorem A.1) this shows the equality (1.2).

We call a transport η admissible, if $(e_0)_\# \eta = \varphi^+$, $(e_1)_\# \eta = \varphi^-$. The construction used to prove (1.4) shows also the existence of an admissible transport η' such that

$$W_1(\varphi^+, \varphi^-) = \int_{\Theta(E)} \ell(\theta) d\eta'(\theta),$$

while using (1.5) for $T = T_\eta$ for an arbitrary admissible transport η and employing Theorem 4.2, we have

$$W_1(\varphi^+, \varphi^-) \leq \int_{\Theta(E)} \ell(\theta) d\eta(\theta),$$

so that in fact we have that $W_1(\varphi^+, \varphi^-)$ is also equal to the minimum among all admissible transports η of the functional $\eta \mapsto \int_{\Theta(E)} \ell(\theta) d\eta(\theta)$. In this way one proves that the representation claimed in Theorem 5.1 is true for optimal (i.e. mass minimizing) currents, and thus all this machinery avoiding the use of representation Theorem 5.1 in a sense amounts to proving it “manually” only for such currents, which are of course automatically acyclic. Thus, once proven for all acyclic currents, Theorem 5.1 becomes an easy and natural alternative to such a machinery (observe that our proof of Theorem A.1 is just few lines). Moreover, a similar result can be proven almost identically with the help of Theorem 5.1 for so-called branched optimal transportation (see [3] for the introduction to the subject), which however goes beyond the purely illustrative scope of this section.

APPENDIX B. METRIC CURRENTS

Throughout the paper we are extensively using the notion of currents with finite mass due to Ambrosio and Kirchheim [1].

For a metric space E we denote

$$D^k(E) := \{(f, \pi_1, \dots, \pi_k) : f \in \text{Lip}_b(E), \pi \in \text{Lip}(E; \mathbb{R}^k)\}.$$

The k -tuples $\omega = (f, \pi_1, \dots, \pi_k) \in D^k(E)$ will be further called k -dimensional differential forms. For the form $\omega = (f, \pi_1, \dots, \pi_k) \in D^k(E)$ we will adopt the notation

$$\omega = f d\pi_1 \wedge d\pi_2 \wedge \dots \wedge d\pi_k = f d\pi.$$

The exterior derivative operator $d: D^k(E) \rightarrow D^{k+1}(E)$ is then defined by

$$d(f d\pi_1 \wedge d\pi_2 \wedge \dots \wedge d\pi_k) := 1 df \wedge d\pi_1 \wedge d\pi_2 \wedge \dots \wedge d\pi_k.$$

Also, given an arbitrary Lipschitz map $\phi: F \rightarrow E$, with F metric space, one defines the pull-back operator $\phi^\#: D^k(E) \rightarrow D^k(F)$ by setting

$$\phi^\#(f d\pi) := f \circ \phi d\pi \circ \phi.$$

Definition B.1. A functional $T: D^k(E) \rightarrow \mathbb{R}$ is called *real k -dimensional metric current* (called further for simplicity *current*) over E , if the following conditions hold:

- (linearity) $(f, \pi_1, \dots, \pi_k) \mapsto T(f, \pi_1, \dots, \pi_k)$ is multilinear, i.e. linear in f and in each of π_i , $i = 1, \dots, k$,
- (continuity) $T(f d\pi_\nu) \rightarrow T(f d\pi)$ whenever $\pi_\nu \rightarrow \pi$ pointwise in $\text{Lip}(E; \mathbb{R}^k)$, as $\nu \rightarrow \infty$, and have uniformly bounded Lipschitz constants,
- (locality) $T(f d\pi) = 0$ whenever for some $i \in \{1, \dots, k\}$ the function π_i is constant in the neighborhood of $\{f \neq 0\}$,
- (finite mass) one has for some finite positive Borel measure μ over E the estimate

$$(2.1) \quad |T(f d\pi)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int_E |f| d\mu$$

valid for every $f \in \text{Lip}_b(E)$, $\pi \in \text{Lip}(E, \mathbb{R}^k)$, with the convention

$$\prod_{i=1}^k \text{Lip}(\pi_i) := 1,$$

if $k = 0$.

The mass measure μ_T is defined to be the minimum over all finite Borel measures μ satisfying (2.1), and the total mass of a current T is defined by $\mathbb{M}(T) := \mu_T(E)$. The support $\text{supp } T$ of a real k -dimensional metric current T with finite mass is defined as the support of μ_T . The set of such currents will be denoted by $\mathcal{M}_k(E)$. The mass functional \mathbb{M} is easily seen to define a norm in $\mathcal{M}_k(E)$.

We will say that a sequence of currents $\{T_\nu\} \subset \mathcal{M}_k(E)$ converges weakly to a current $T \in \mathcal{M}_k(E)$, and write $T_\nu \rightharpoonup T$, if $T_\nu(\omega) \rightarrow T(\omega)$ as $\nu \rightarrow \infty$, for every $\omega \in D^k(E)$. The mass is known to be lower semicontinuous with respect to weak convergence of currents [1].

Clearly, every metric current $T \in \mathcal{M}_k(E)$ may be extended by continuity from the space of forms $D^k(E)$ to the larger space of $(k+1)$ -tuples (f, π_1, \dots, π_k) , where $\pi \in \text{Lip}(E; \mathbb{R}^k)$, while $f: E \rightarrow \mathbb{R}$ is a bounded Borel function on E . Thus, whenever E is a complete metric space, then every $T \in \mathcal{M}_0(E)$ is represented by some signed Borel measure of finite total variation over E (given by the set function $B \mapsto T(1_B)$ for every Borel set $B \subset E$, where 1_B stands for the characteristic function of B). Hence, when necessary, we will always identify a $T \in \mathcal{M}_0(E)$ with the respective signed measure. Note that the mass \mathbb{M} over $\mathcal{M}_0(E)$ is nothing else than the total variation norm $\|\cdot\|$ over the space of such measures on E .

If $T \in \mathcal{M}_k(E)$ and $\omega = g d\tau \in D^m(E)$, $m \leq k$, we define the restricted metric current $T \llcorner \omega \in \mathcal{M}_{k-m}(E)$ by the formula

$$T \llcorner \omega(f d\pi) := T(fg, \tau_1, \dots, \tau_m, \pi_1, \dots, \pi_{k-m}) \text{ for all } f d\pi \in D^{k-m}(E).$$

Since T is assumed to have finite mass, then in the above formula one may admit in place of f and g any bounded Borel functions. In particular, whenever $\omega = 1_B \in D^0(E)$ for some Borel set $B \subset E$, we will simply write $T \llcorner B$ for $T \llcorner \omega$.

The boundary ∂T of a k -dimensional current T is a $(k-1)$ -dimensional current defined by the formula

$$\partial T(\omega) := T(d\omega) \text{ for all } \omega \in D^{k-1}(E).$$

Further, for an arbitrary Lipschitz map $\phi: F \rightarrow E$, with F metric space, we define the push-forward operator $\phi_\#: \mathcal{M}_k(F) \rightarrow \mathcal{M}_k(E)$ on currents by setting

$$(\phi_\# T)(\omega) := T(\phi^* \omega) \text{ for all } \omega \in D^k(E).$$

We say that T is a *normal current*, if $\mathbb{M}(T) < +\infty$ and $\mathbb{M}(\partial T) < +\infty$. It is worth remarking that in a finite-dimensional Euclidean space $E = \mathbb{R}^n$ every normal current (in the sense of metric currents) by theorem 11.1 from [1] may be identified via a natural isomorphism with a Whitney normal current.

If E is a normed space, we call oriented segment $\llbracket a, b \rrbracket$ the curve θ (or, to be more precise, the equivalence class of curves in $\Theta(E)$) that may be parameterized by $\theta(t) := (1-t)a + tb$, $t \in [0, 1]$. We identify oriented segments with one-dimensional currents associated with them. We further call $T \in \mathcal{M}_1(E)$ *polyhedral* current, if $T = \sum_\nu \theta_\nu T_\nu$, where $\theta_\nu > 0$, and T_ν are currents associated to oriented segments $T_\nu = \llbracket a_\nu, b_\nu \rrbracket$ which may overlap only at the endpoints.

The following easy statement regarding weak convergence of metric currents has to be mentioned.

Lemma B.2. *Let $T_\nu \in \mathcal{M}_1(E)$, $T_\nu \rightharpoonup T$ in the weak sense of currents and $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$. Then $\mu_{T_\nu} \rightharpoonup \mu_T$ in the narrow sense of measures and in particular, the sequence of measures $\{\mu_{T_\nu}\}$ is uniformly tight.*

Remark B.3. The conclusion on uniform tightness of $\{\mu_{T_\nu}\}$ is true by theorem 8.6.4 from [4] if the measures μ_{T_ν} and μ_T are tight (which is automatically satisfied once one makes the set-theoretical assumption mentioned in the Introduction).

Proof. One has $\mu_{T_\nu}(E) \rightarrow \mu(E)$ and

$$\mu_T(U) \leq \liminf_{\nu} \mu_{T_\nu}(U)$$

for every open $U \subset E$, and therefore $\mu_{T_\nu} \rightarrow \mu_T$ in the narrow sense of measures by theorem 8.2.3 from [4]. The uniform tightness of $\{\mu_{T_\nu}\}$ follows then from Prokhorov theorem for nonnegative measures (theorem 8.6.4 from [4]). \square

APPENDIX C. POLYHEDRAL APPROXIMATION IN FINITE DIMENSIONS

This section contains an auxiliary assertion on approximation of currents over a finite-dimensional normed space E . In the case when $E = \mathbb{R}^n$ is Euclidean, analogous results can be found, e.g., in [5][4.1.23, 4.2.24] (our result is a bit more precise for one-dimensional currents and tailored for our purposes so as to be used in combination with Lemma 5.3 and Proposition 4.6). Throughout this section $\dim E = n$, and E is assumed to be endowed with some norm $\|\cdot\|$, while the notation \mathbb{R}^n will stand for the same space endowed with the Euclidean norm $|\cdot|$. We denote by $\mathcal{F}(T)$ the *flat norm* of T defined by

$$\mathcal{F}(T) := \inf\{\mathbb{M}(A) + \mathbb{M}(B) : A \in \mathcal{M}_k(E), B \in \mathcal{M}_{k+1}(E), A + \partial B = T\}.$$

Lemma C.1. *Let $T \in \mathcal{M}_1(E)$ be a normal current with compact support over the finite-dimensional space E , and $r > 0$ be such that $\text{supp } T \subset B_r(0) \subset E$. Then there is a sequence of one-dimensional real polyhedral currents T_ν over E with $\text{supp } T_\nu \subset B_{2r}(0)$, which converge to T in the flat norm (in particular, weakly), i.e. $\mathcal{F}(T_\nu - T) \rightarrow 0$, while $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ in the $*$ -weak sense of measures (in particular, $\mathbb{M}(\partial T_\nu) \rightarrow \mathbb{M}(\partial T)$) and $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$. If T is acyclic, one may choose T_ν to be acyclic too.*

Proof. Step 1. We first show adapting the proof of [5][4.1.23] that T may be approximated in flat norm by a sequence of polyhedral currents $S_\nu \in \mathcal{M}_1(E)$ supported over $B_r(0)$ with $\mathbb{M}(S_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$. For this purpose, first, by choosing the approximate identity

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right),$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geq 0$, $\|\varphi\|_1 = 1$, $\varepsilon > 0$, define $T_\varepsilon \in \mathcal{M}_1(E)$ by setting

$$T_\varepsilon(\omega) := \int_{\mathbb{R}^n} (\tau_{x\#} T)(\omega) \varphi_\varepsilon(-x) dx$$

for all $\omega \in D^1(E)$. Once one considers T and T_ε as currents over \mathbb{R}^n (with Euclidean norm), by [5][4.1.18] one gets $\mathcal{F}_2(T_\varepsilon - T) \leq \varepsilon(\mathbb{M}_2(T) + \mathbb{M}_2(\partial T))$, where \mathcal{F}_2 and \mathbb{M}_2 stand for the flat norm and mass over Euclidean flat chains. Hence $\mathcal{F}(T_\varepsilon - T) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. On the other hand,

$$\mathbb{M}(T_\varepsilon) \leq \int_{\mathbb{R}^n} \mathbb{M}(\tau_{x\#} T) \varphi_\varepsilon(-x) dx = \int_{\mathbb{R}^n} \mathbb{M}(T) \varphi_\varepsilon(-x) dx = \mathbb{M}(T),$$

which combined with lower semicontinuity of mass gives $\mathbb{M}(T_\varepsilon) \rightarrow \mathbb{M}(T)$ as $\varepsilon \rightarrow 0^+$. Analogously, one has

$$\partial T_\varepsilon(\omega) := \int_{\mathbb{R}^n} (\tau_{x\#} \partial T)(\omega) \varphi_\varepsilon(-x) dx$$

for every $\omega \in D^0(E)$, hence $\mathbb{M}(\partial T_\varepsilon) \rightarrow \mathbb{M}(\partial T)$ as $\varepsilon \rightarrow 0^+$. Also, clearly, $\text{supp } T_\varepsilon \subset B_r(0)$ once $\varepsilon > 0$ is sufficiently small. But

$$T_\varepsilon(f d\pi) = \int_{\mathbb{R}^n} f(x) (\nabla \pi(x), l) dx$$

for some integrable vector field $l = l_\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (cfr. proposition 6.1 combined with theorem 1.3 in [8]), and therefore this reduces the proof of the desired assertion to the case of T having exactly such form.

We may thus assume now $T(f d\pi) := \int_{\mathbb{R}^n} f(x)(\nabla\pi(x), l) dx$ for some integrable vector field $l: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Since $\|\nabla\pi(x)\|' \leq \text{Lip } \pi$ for all $x \in E$, where $\|\cdot\|'$ stands for the norm in the space E' dual to E , one clearly has $\mu_T \leq \|l\| dx$. Moreover, in fact the equality $\mu_T = \|l\| dx$ holds. It is clearly enough to prove this for the case l is a simple (i.e. finite valued) function, that is, $l = \sum_{i=1}^m l_i 1_{E_i}$ for some constants $l_i \in E$ and Borel sets $E_i \subset E$. In this case just take $l'_i \in E'$ be such that $(l'_i, l_i) = \|l_i\|$, $\|l'_i\|' = 1$ and $\pi_i: E \rightarrow \mathbb{R}$ be affine functions such that $\nabla\pi_i = l'_i$, hence $\text{Lip } \pi_i \leq 1$. Then

$$\mu_T(e) = \mathbb{M}(T \llcorner e) \geq \sum_{i=1}^m T(1_{E_i \cap e} d\pi_i) = \sum_{i=1}^m \mathcal{L}^n(E_i \cap e) \|l_i\| = \int_e \|l\| dx,$$

where \mathcal{L}^n stands for the n -dimensional Lebesgue measure in E .

Approximating l by piecewise constant vector fields l_k which are constant over a finite number of rectangles $R_i^k \subset E$, with one side of the rectangle parallel to the direction of l_k inside R_i^k , the approximation being intended in the sense $\int_{\mathbb{R}^n} |l - l_k| dx \rightarrow 0$ as $k \rightarrow +\infty$ (so that the currents T_k defined by $T_k(f d\pi) := \int_{\mathbb{R}^n} f(x)(\nabla\pi(x), l_k) dx$, converge to T in mass), we reduce the problem to the case

$$T(f d\pi) := \int_R f(x)(\nabla\pi(x), l) dx,$$

where $R \subset \mathbb{R}^n$ is a rectangle and $l(x)$ is constant and parallel to one of the sides of R . Let $[a, b]$ be a side of R parallel to l and directed in the same direction as l (i.e. with the vector $b - a$ having the direction of l), and consider the $(n-1)$ -dimensional face S of R perpendicular to l such that $a \in S$. Dividing S by a uniform dyadic grid with nodes $\{x_i\}_{i=1}^{N_\nu}$, with $N_\nu = 2^\nu - 1$, and setting $\theta_i(t) := x_i + tl$ for $t \in [0, 1]$, we let

$$S_\nu := \frac{\mathcal{L}^n(R) \cdot \|l\|}{N_\nu \|b - a\|} \sum_{i=1}^{N_\nu} \llbracket \theta_i \rrbracket,$$

so that, minding $\mathbb{M}(\llbracket \theta_i \rrbracket) = \ell(\theta_i) = \|b - a\|$, we have $\mathbb{M}(S_\nu) = \mathcal{L}^n(R) \cdot \|l\| = \mathbb{M}(T)$. Clearly, one has $\mathcal{F}(S_\nu - T) \rightarrow 0$ as $\nu \rightarrow \infty$ (e.g. one may refer to the fact that $\mathbb{M}_2(T) = \mathcal{L}^n(R) \cdot \|l\| = \mathbb{M}_2(S_\nu)$, and $\mathcal{F}_2(S_\nu - T) \rightarrow 0$ as $\nu \rightarrow \infty$).

Step 2. Let S_ν be a sequence constructed in Step 1 of the proof. We as usual identify finite purely atomic measures with zero-dimensional polyhedral currents. Recall that $(\partial T)^+$ has the same total mass as $(\partial T)^-$ since

$$\partial((\partial T)^+ - (\partial T)^-) = \partial\partial T = 0.$$

Let ϕ_ν^\pm be purely atomic measures with compact support over $B_r(0)$, having the same total mass as $(\partial T)^\pm$ (so that in particular, $\mathbb{M}(\phi_\nu^+ - \phi_\nu^-) = \mathbb{M}(\partial T)$) and such that

$$\mathcal{F}(\phi_\nu^\pm - (\partial T)^\pm) \rightarrow 0$$

as $\nu \rightarrow \infty$ (recall that in fact, \mathcal{F}_2 , and hence also \mathcal{F} , metrizes $*$ -weak topology over the set of finite Borel measures concentrated over a compact subset of E). We now act as in the proof of [5][4.2.24]. Mind that $\mathcal{F}(\partial T - \partial S_\nu) \leq \mathcal{F}(T - S_\nu)$ and hence

$$\mathcal{F}(\phi_\nu^+ - \phi_\nu^- - \partial S_\nu) \leq \mathcal{F}(\phi_\nu^+ - \phi_\nu^- - \partial T) + \mathcal{F}(T - S_\nu) \rightarrow 0$$

as $\nu \rightarrow \infty$. Using [5][4.2.23] choose now one-dimensional polyhedral currents Y_ν with $\text{supp } Y_\nu \subset B_r(0)$ such that

$$\mathbb{M}(\phi_\nu^+ - \phi_\nu^- - \partial S_\nu - \partial Y_\nu) + \mathbb{M}(Y_\nu) \rightarrow 0,$$

and set $T_\nu := S_\nu + Y_\nu$, so that $\text{supp } T_\nu \subset B_{2r}(0)$. One has then

$$\begin{aligned} |\mathbb{M}(T_\nu) - \mathbb{M}(T)| &\leq |\mathbb{M}(S_\nu) - \mathbb{M}(T)| + \mathbb{M}(Y_\nu) \rightarrow 0, \\ |\mathbb{M}(\partial T_\nu) - \mathbb{M}(\partial T)| &= |\mathbb{M}(\partial S_\nu + \partial Y_\nu) - \mathbb{M}(\partial T)| \\ &\leq \mathbb{M}(\phi_\nu^+ - \phi_\nu^- - \partial S_\nu - \partial Y_\nu) + |\mathbb{M}(\phi_\nu^+ - \phi_\nu^-) - \mathbb{M}(\partial T)| \\ &= \mathbb{M}(\phi_\nu^+ - \phi_\nu^- - \partial S_\nu - \partial Y_\nu) \rightarrow 0, \end{aligned}$$

while

$$\mathcal{F}(T_\nu - T) \leq F(T_\nu - T) + \mathcal{F}(Y_\nu) \leq F(T_\nu - T) + \mathbb{M}(Y_\nu) \rightarrow 0$$

as $\nu \rightarrow \infty$. Finally, viewing ∂T_ν and ∂T as signed measures, we have that the total variations of the former are uniformly bounded and converge to that of the latter, and therefore $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ in the $*$ -weak sense of measures as $\nu \rightarrow \infty$ by corollary 8.4.8 of [4].

Step 3. If T is acyclic, we modify T_ν in the following way. Let C_ν be the cycle of T_ν given by Proposition 3.8 such that $T'_\nu := T_\nu - C_\nu$ is acyclic. Up to a subsequence (not relabeled), $C_\nu \rightarrow C$ as $\nu \rightarrow \infty$. Hence, by Lemma 3.6, $\mathbb{M}(C_\nu) \rightarrow \mathbb{M}(C)$ as $\nu \rightarrow \infty$ and C is a cycle of T . Since the only cycle of T is zero we conclude that $\mathbb{M}(C_\nu) \rightarrow 0$, which means that $T'_\nu \rightarrow T$ and $\mathbb{M}(T'_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$.

It remains to observe that $T'_\nu \leq T_\nu$, and since $T_\nu = \sum_{i=1}^{m_\nu} \alpha_{i,\nu} \llbracket \Delta_{i,\nu} \rrbracket$, where $\alpha_{i,\nu} \in \mathbb{R}$ and $\Delta_{i,\nu} \subset E$ are segments which may overlap only at the endpoints, then

$$(3.1) \quad T'_\nu \llcorner \Delta_{i,\nu} \leq T_\nu \llcorner \Delta_{i,\nu}$$

by Remark 3.5 for all $i = 1, \dots, m_\nu$. Further, one has

$$(3.2) \quad \partial(T'_\nu \llcorner \Delta_{i,\nu}) \leq \partial(T_\nu \llcorner \Delta_{i,\nu})$$

for all $i = 1, \dots, m_\nu$, since otherwise by Lemma C.2 one would have that $\partial(T'_\nu \llcorner \Delta_{i,\nu})$ charges the interior of a segment $\Delta_{i,\nu}$ for some $i = 1, \dots, m_\nu$, which would contradict $\partial T'_\nu = \partial T_\nu$ (the latter measure being concentrated only at the endpoints of the segments $\Delta_{i,\nu}$). Therefore, from (3.1) and (3.2) with the help of Lemma C.2 we get

$$T'_\nu \llcorner \Delta_{i,\nu} = \beta_{i,\nu} \llbracket \Delta_{i,\nu} \rrbracket$$

for some $\beta_{i,\nu} \in \mathbb{R}$, and hence T'_ν is still polyhedral. \square

Lemma C.2. *Let E be an arbitrary metric space. If $T = \sigma_\# \llbracket a, b \rrbracket$, where σ is injective, $\{a, b\} \in \mathbb{R}$ and $S \leq T$, then $S = T \llcorner \lambda$ for some Borel function $\lambda: E \rightarrow [0, 1]$. Further, either $\partial S \leq \partial T$, which happens if and only if $\lambda \in [0, 1]$ is constant over $\text{supp } T$, or ∂S considered as a measure charges $\sigma([a, b]) \setminus \{\sigma(a), \sigma(b)\}$.*

Proof. Assume without loss of generality that σ is parameterized by arclength (in particular, then $a = 0$), so that σ is an isometry between $[a, b]$ and $\sigma([a, b])$. Denote

$$\tilde{S} := \sigma_\#^{-1} S, \quad \tilde{T} := \llbracket a, b \rrbracket,$$

so that in particular

$$\tilde{T} - \tilde{S} = \sigma_\#^{-1}(T - S).$$

Since $S \leq T$, then by Remark 3.5 one has $\mu_S \leq \mu_T$ and hence $\mu_S = \lambda \mu_T$ for some Borel function λ satisfying $0 \leq \lambda \leq 1$. Minding now that σ is an isometry, we get

$$\begin{aligned} \mu_{\tilde{S}} &= \sigma_\#^{-1} \mu_S = (\lambda \circ \sigma) \\ \mu_{\tilde{T} - \tilde{S}} &= \sigma_\#^{-1} \mu_{T - S} = (1 - \lambda \circ \sigma) \mu_{\tilde{T}}, \end{aligned}$$

where $\mu_{\tilde{T}} = \mathcal{L}^1 \llcorner [a, b]$. This means $\mu_{\tilde{S}} + \mu_{\tilde{T} - \tilde{S}} = \mu_{\tilde{T}}$, or, in other words, $\tilde{S} \leq \tilde{T}$. Now, since \tilde{S} and \tilde{T} are one-dimensional currents in \mathbb{R} , then $\tilde{S} = \tilde{T} \llcorner \alpha$ for some Borel function α satisfying $0 \leq \alpha \leq 1$. Therefore, $\alpha = \lambda \circ \sigma$, which implies $S = T \llcorner \lambda$. Analogously one gets $\partial \tilde{S} \leq \partial \tilde{T} = \delta_a - \delta_b$, which is only possible if α is constant over $[a, b]$ (minding that $\tilde{S} = \llbracket a, b \rrbracket \llcorner \alpha$). Hence, also λ is constant, and this completes the proof. \square

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